The Pure Pattern Type Systems

Semantics

Type systems

Strong normalization

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History

- **Combinations of rewriting and lambda-calculus**
  - 1980 CRS
  - 1990 Combination of lambda calculus and rewriting
  - 1990 Higher-order rewriting systems
  - 1990 Rewriting logic
  - 2000 Combination of CC and rewriting
- **The ρ-calculus** 1999
  - confluence (evaluation strategies) and termination (simply typed calculus)
  - operational semantics of ELAN
- **Rho-calculus** 2001
  - representation of object calculi (ObjCal and LambdaObjCal)
  - sophisticated type systems - the Rho-cube

Why a new calculus?

Rewriting is nice, but

- the rewrite relation is difficult to control
- non-reducibility is impossible to express
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Lambda-calculus is great, but

- lacks of discrimination capabilities
- difficult to control

A calculus with more explicit features

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In “basic” PPTS,

- rules and results are first class objects
- application, decision of redex reduction are explicit
- the matching theory is a main explicit parameter

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In “basic” PPTS,

- rules and results are first class objects
- application, decision of redex reduction are explicit
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This allows for advanced calculi,

- with explicit constraints and substitutions
- to even dissociate binding from matching when abstracting

The “untyped” PPTS

Expressiveness of the calculus

Typed recursion

Strong normalization

The Untyped Syntax

$$\mathcal{P} \;::=\; \mathcal{T} \qquad\qquad \text{Patterns}$$

$$\mathcal{T} \;::=\; \mathcal{X} \mid \mathcal{K} \mid \boldsymbol{\lambda}\mathcal{P}.\mathcal{T} \mid \mathcal{T}\bullet\mathcal{T} \mid [\mathcal{P} \ll \mathcal{T}]\mathcal{T} \mid \mathcal{T};\mathcal{T} \qquad \text{Terms}$$

1. $\boldsymbol{\lambda}T_1.T_2$ denotes a *rule abstraction* with pattern T_1 and body T_2
   ... the free variables of T_1 are bound in T_2

2. $[T_1 \ll T_2]T_3$ denotes a *delayed matching constraint*
   ... the free variables of T_1 are bound in T_3 but not in T_2

3. The terms can be also *structures* built using the symbol ";"

4. We work modulo the *α-convention* and the *hygiene-convention*

Some ρ-terms

$(\boldsymbol{\lambda}\mathcal{X}.\mathcal{X})\bullet a$ similar to the λ-term $(\lambda x.x)\ a$

$(\boldsymbol{\lambda}\mathcal{X}.\mathcal{X}\bullet\mathcal{X})\bullet(\boldsymbol{\lambda}\mathcal{X}.\mathcal{X}\bullet\mathcal{X})$ the well-known λ-term $(\omega\omega)$

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$(\boldsymbol{\lambda}a.b)\bullet a$ the application of the rule $a \rightarrow b$ to the term a

$(\boldsymbol{\lambda}f(\mathcal{X},\mathcal{Y}).g(\mathcal{X},\mathcal{Y}))\bullet f(a,b)$ a classical rewrite rule application

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$(\boldsymbol{\lambda}f(\mathcal{X},\mathcal{Y}).g(\mathcal{X},\mathcal{Y}))^{\bullet}f(a,b)$ a classical rewrite rule application

$(\boldsymbol{\lambda}a.b;\boldsymbol{\lambda}a.c)^{\bullet}a$ "non-deterministic" application of two rules

The Small-step Reduction Semantics

$$(\boldsymbol{\lambda} P.A)\bullet B \;\to_\rho\; [P \ll B]A$$

$$[P \ll B]A \;\to_\sigma\; A\theta \qquad \text{with } P\theta = B$$

$$(A; B)\bullet C \;\to_\delta\; A\bullet C; B\bullet C$$

Intuition on the small-step Semantics

$$\begin{array}{lll} (\boldsymbol{\lambda} P.A)^{\bullet}B & \to_{\rho} & [P \ll B]A \\ & \to_{\sigma} & A\theta \\ & & \quad \text{if } P\theta =_{\mathbb{T}} B \end{array}$$

$$\begin{array}{lll} (\boldsymbol{\lambda} P.A)^{\bullet}B & \to_{\rho} & [P \ll B]A \\ & & \text{STOP!} \\ & & \quad \text{if } \not\exists\theta.\ P\theta =_{\mathbb{T}} B \end{array}$$

With suitable restrictions on the shape of patterns $\mathcal{P}$, the calculus is confluent.

Intuition on the small-step Semantics

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With suitable restrictions on the shape of patterns $\mathcal{P}$, the calculus is confluent.

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$(\boldsymbol{\lambda}\mathcal{X}.\mathcal{X})^{\bullet}a$

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$(\boldsymbol{\lambda}f(\mathcal{X},\mathcal{Y}).g(\mathcal{X},\mathcal{Y}))^{\bullet}(f(a,b))$

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$$(\boldsymbol{\lambda}\mathcal{X}.\mathcal{X})^{\bullet}a \quad \mapsto\!\!\!\rightarrow_{\rho\sigma\delta} a$$

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$(\boldsymbol{\lambda}\mathcal{X}.(\mathcal{X}^{\bullet}\mathcal{X}))^{\bullet}(\boldsymbol{\lambda}\mathcal{X}.(\mathcal{X}^{\bullet}\mathcal{X})) \quad \longmapsto\!\!\!\!\rightarrow_{\rho\sigma\delta} \{\omega^{\bullet}\omega\} \longmapsto\!\!\!\!\rightarrow_{\rho\sigma\delta} \ \ldots$

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$$(\boldsymbol{\lambda}a.b; \boldsymbol{\lambda}a.c)^{\bullet}a \quad \mapsto\!\!\!\!\to_{\rho\sigma\delta} (\boldsymbol{\lambda}a.b)^{\bullet}a; (\boldsymbol{\lambda}a.c)^{\bullet}a \mapsto\!\!\!\!\to_{\rho\sigma\delta} b; c$$

Simple Success Reduction

$$
\begin{array}{rcl}
(\boldsymbol{\lambda} f(\mathcal{X}).(\boldsymbol{\lambda} 3.3)\bullet\mathcal{X})\bullet f(3) & \mapsto_\rho & [f(\mathcal{X}) \ll f(3)]((\boldsymbol{\lambda} 3.3)\bullet\mathcal{X}) \\
& \mapsto_\sigma & (\boldsymbol{\lambda} 3.3)\bullet 3 \\
& \mapsto_\rho & [3 \ll 3]3 \\
& \mapsto_\sigma & 3
\end{array}
$$

$$
\begin{array}{rcl}
(\boldsymbol{\lambda} f(\mathcal{X}).(\boldsymbol{\lambda} 3.3)\bullet\mathcal{X})\bullet f(3) & \mapsto_\rho & (\boldsymbol{\lambda} f(\mathcal{X}).[3 \ll \mathcal{X}]3)\bullet f(3) \\
& \mapsto_\rho & [f(\mathcal{X}) \ll f(3)]([3 \ll \mathcal{X}]3) \\
& \mapsto_\sigma & [3 \ll 3]3 \\
& \mapsto_\sigma & 3
\end{array}
$$

Simple Failure Reduction

$$
\begin{array}{lll}
(\boldsymbol{\lambda} f(\mathcal{X}).(\boldsymbol{\lambda} 3.3)^{\bullet}\mathcal{X})^{\bullet} f(4) & \mapsto_{\rho} & [f(\mathcal{X}) \ll f(4)]((\boldsymbol{\lambda} 3.3)^{\bullet}\mathcal{X}) \\
& \mapsto_{\sigma} & (\boldsymbol{\lambda} 3.3)^{\bullet} 4 \\
& \mapsto_{\rho} & [3 \ll 4]3 \\
(\boldsymbol{\lambda} f(\mathcal{X}).(\boldsymbol{\lambda} 3.3)^{\bullet}\mathcal{X})^{\bullet} f(4) & \mapsto_{\rho} & (\boldsymbol{\lambda} f(\mathcal{X}).[3 \ll \mathcal{X}]3)^{\bullet} f(4) \\
& \mapsto_{\rho} & [f(\mathcal{X}) \ll f(4)]([3 \ll \mathcal{X}]3) \\
& \mapsto_{\sigma} & [3 \ll 4]3
\end{array}
$$

The “untyped” PPTS

Expressiveness of the calculus

Typed recursion

Strong normalization

About the expressiveness of the PPTS in $\mathbb{T}_\emptyset$

1. **Compiling the Λ into the PPTS.** $\varphi : \Lambda \Rightarrow PPTS$

   (a) $\varphi(X) = X$
   (b) $\varphi(\lambda X.M) = \boldsymbol{\lambda} X.\varphi(M)$
   (c) $\varphi(M\ N) = \varphi(M)\bullet\varphi(N)$

   Theorem: $M \mapsto_\beta^n N$ if and only if $\varphi(M) \mapsto\!\!\!\to_{\rho\sigma\delta}{}^{2n} \varphi(N)$

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2. **Encoding Rewriting**

   (a) A rewrite system $\mathcal{R}$ can be represented as the structure containing all the rules
   (b) Reduction strategies can be encoded
   Theorem: If $T_1 \mapsto_\mathcal{R} T_2$, then $\exists\ T_\mathcal{R}$ such that $T_\mathcal{R}\bullet T_1 \mapsto\!\!\!\!\to_{\rho\sigma\delta} T_2$

Expressiveness in other theories

3. **Encoding object calculi**

   (a) Lambda-calculus of objects, using associative ";"

   (b) Object Calculus, using associative and commutative ";"

Expressiveness in other theories

3. **Encoding object calculi**
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4. **Encoding Rewriting**
   (a) A rewrite system $\mathcal{R}$ can be represented as the structure containing all the rules
   (b) Reduction strategies can be encoded
   (c) A special theory is used for detecting and erasing matching failures

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A Simple Type System

$$\frac{\mathcal{X} : \sigma \in \Gamma}{\Gamma \vdash \mathcal{X} : \sigma} \; (Var)$$

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$$\frac{\Gamma, \Delta \vdash \mathcal{T}_1 : \sigma \quad \Gamma, \Delta \vdash \mathcal{T}_2 : \tau}{\Gamma \vdash \boldsymbol{\lambda}(\mathcal{T}_1{:}\Delta).\mathcal{T}_2 : \sigma \rightarrow \tau} \ (Abs)$$

Typing properties

Well-typed matching:
If $\mathcal{S}ol(\mathcal{P} \prec\!\!\prec \mathcal{T}) = \theta$, then $\forall X \in \mathcal{P}, \quad \Gamma \vdash X : \sigma \;\Rightarrow \Gamma \vdash X\theta : \sigma$.

Subject Reduction:
If $\Gamma \vdash \mathcal{T}_1 : \sigma$ and $\mathcal{T}_1 \mapsto\!\!\!\!\rightarrow_{\rho\sigma\delta} \mathcal{T}_2$, then $\Gamma \vdash \mathcal{T}_2 : \sigma$.

Uniqueness:
If $\Gamma \vdash \mathcal{T} : \varphi$ and $\Gamma \vdash \mathcal{T} : \psi$, then $\varphi =_\alpha \psi$.

Decidability:

$$\left.\begin{array}{ll} \textit{(typechecking)} & \Gamma \vdash \mathcal{T} : \varphi\,? \\ \textit{(type inference)} & \Gamma \vdash \mathcal{T} : \,? \end{array}\right\} \text{are decidable.}$$

Normalization failure

$f : (\alpha \rightarrow\!\!\triangleright \alpha) \rightarrow\!\!\triangleright \alpha$ and $\Gamma = X : \alpha \rightarrow\!\!\triangleright \alpha$, $\qquad \omega \triangleq \boldsymbol{\lambda} f(X).X \bullet (f \bullet X)$

$$
\begin{array}{lcl}
\omega \bullet (f \bullet \omega) & \equiv & (\boldsymbol{\lambda} f(X) \,.\, X \bullet (f \bullet X)) \bullet (f \bullet \omega) \\
& \mapsto_\rho & [f(X) \ll f(\omega)].(X \bullet (f \bullet X)) \\
& \mapsto_\sigma & \omega \bullet (f \bullet \omega) \\
& \mapsto_\rho & \ldots
\end{array}
$$

Normalization failure (cont'd)

$$f : (\alpha \rightarrow \alpha) \rightarrow \alpha \text{ and } \Gamma = X : \alpha \rightarrow \alpha, \qquad \omega \triangleq \boldsymbol{\lambda} f(X).X \bullet (f \bullet X)$$

$$\dfrac{\dfrac{\Gamma \vdash f : (\alpha \rightarrow \alpha) \rightarrow \alpha \quad \Gamma \vdash X : \alpha \rightarrow \alpha}{\Gamma \vdash f(X) : \alpha} \quad \dfrac{\Gamma \vdash X : \alpha \rightarrow \alpha \quad \dfrac{\vdots}{\Gamma \vdash f \bullet X : \alpha}}{\Gamma \vdash X \bullet (f \bullet X) : \alpha}}{\vdash \omega \equiv \boldsymbol{\lambda} f(X).X \bullet (f \bullet X) : \alpha \rightarrow \alpha}$$

$$\vdash \omega \bullet (f \bullet \omega) : \alpha$$

Logical inconsistency

- As is, the Curry-Howard isomorphism is not valid:

$$\frac{\Gamma, \Delta \vdash \mathcal{T}_1 : \sigma \quad \Gamma, \Delta \vdash \mathcal{T}_2 : \tau}{\Gamma \vdash \boldsymbol{\lambda}(\mathcal{T}_1{:}\Delta)\ .\ \mathcal{T}_2 : \sigma \rightarrow \tau}\ (Abs) \qquad \frac{\Gamma, \Delta \vdash \sigma \quad \Gamma, \Delta \vdash \tau}{\Gamma \vdash \sigma \rightarrow \tau}\ (\rightarrow I)$$

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- How to fix it ?

$$\frac{\Gamma, X_i : \varphi_i \vdash B : \psi}{\Gamma \vdash \boldsymbol{\lambda} A.B : (\bigwedge \varphi_i) \rightarrow \psi}\ (Abs) \quad , \quad FV(A) = \{X_i^{\varphi_i}\}$$

But what will (App) become ?

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Strong normalization

Dependent type discipline

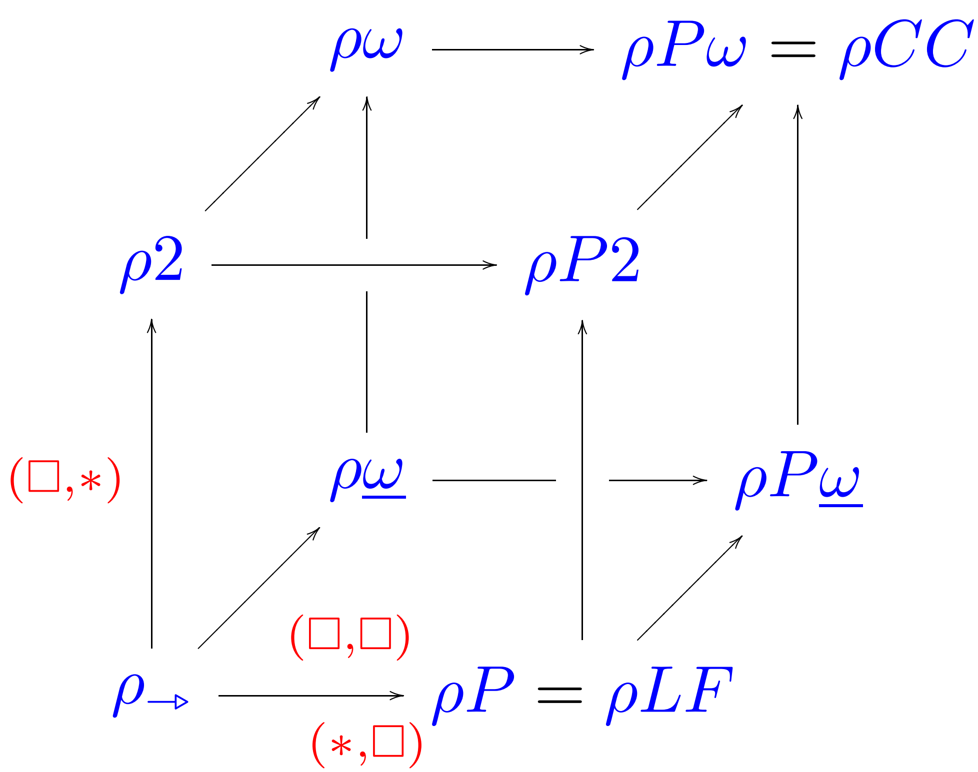
$$\frac{\Gamma, \Delta \vdash \mathcal{T}_1 : \sigma \quad \Gamma, \Delta \vdash \mathcal{T}_2 : \tau}{\Gamma \vdash \boldsymbol{\lambda}(\mathcal{T}_1{:}\Delta) \,.\, \mathcal{T}_2 : \sigma \rightarrowtriangle \tau} \; (Abs)$$

$$\frac{\Gamma \vdash \mathcal{T}_1 : \sigma \rightarrowtriangle \tau \quad \Gamma \vdash \mathcal{T}_2 : \sigma}{\Gamma \vdash \mathcal{T}_1 \; \mathcal{T}_2 : \tau} \; (Appl)$$

$$\frac{\Gamma, \Delta \vdash \mathcal{T}_2 : \tau \quad \Gamma \vdash \boldsymbol{\Pi}(\mathcal{T}_1{:}\Delta) \,.\, \tau : s}{\Gamma \vdash \boldsymbol{\lambda}(\mathcal{T}_1{:}\Delta) \,.\, \mathcal{T}_2 : \boldsymbol{\Pi}(\mathcal{T}_1{:}\Delta) \,.\, \tau} \; (Abs)$$

$$\frac{\Gamma \vdash \mathcal{T}_1 : \boldsymbol{\Pi}(\mathcal{T}_{11}{:}\Delta) \,.\, \tau \quad \Gamma \vdash \mathcal{T}_2 : \sigma \quad \Gamma, \Delta \vdash \mathcal{T}_{11} : \sigma}{\Gamma \vdash \mathcal{T}_1 \; \mathcal{T}_2 : [\mathcal{T}_{11} \ll_\Delta \mathcal{T}_2].\tau} \; (Appl)$$

The ρ-cube


$\rho\omega$
$\rho P\omega = \rho CC$
$\rho 2$
$\rho P2$
$(\square,*)$
$\rho\underline{\omega}$
$\rho P\underline{\omega}$
$(\square,\square)$
$\rho_{\rightarrow}$
$\rho P = \rho LF$
$(*,\square)$

Typing properties

- Substitution, Subject reduction: still OK.
- Correctness: $\vdash A : B \;\Rightarrow\; \vdash B : s$
- Consistency: $\nvdash A : \bot \;\; (\triangleq \; X^* \rightharpoonup X^*)$
  ($\forall A$ closed, with "honest" constants and in normal form)
- Uniqueness in $\rho 2$
  $\vdash_{\rho 2} A : B \; and \; \vdash_{\rho 2} A : B' \;\Rightarrow\; B =_{\rho\sigma\delta} B'$

Restricted typing

In the enhanced system:

$$\vdash \omega \triangleq \boldsymbol{\lambda} f \bullet X . X \bullet (f \bullet X) : \boldsymbol{\Pi} f \bullet X . \alpha$$

$$\vdash f : \boldsymbol{\Pi}(Y : (\boldsymbol{\Pi} Z . \alpha))) . \alpha$$

But in $f \bullet \omega$ the pattern Y and the argument ω must have a common type:

$$\frac{\Gamma \vdash f : \boldsymbol{\Pi}(Y : \boldsymbol{\Pi} Z . \alpha)) . \alpha \quad \Gamma \vdash \omega : \boldsymbol{\Pi} f \bullet X . \alpha \quad \Gamma, \Delta \vdash Y : \boldsymbol{\Pi} Z . \alpha}{\Gamma \nvdash f \ \omega} \ (Appl)$$

With pattern dependent types, $f \bullet \omega$ (and thus $\omega \bullet (f \bullet \omega)$) is not typable.

Encoding the PPTS into λ-calculus

$$[\![X]\!] \triangleq X$$

$$[\![f_i]\!] \triangleq \lambda x_1 \ldots \lambda x_{\alpha_i}.\ (\lambda z_1 \ldots \lambda z_S.(z_i\ x_1 \ldots x_{\alpha_i}))$$

$$[\![f_i \bullet B_1 \ldots \bullet B_{\alpha_i}]\!] \triangleq \lambda z_1 \ldots \lambda z_S.(z_i\ B_1 \ldots B_{\alpha_i})$$

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[\![f_i \bullet B_1 \ldots \bullet B_{\alpha_i}]\!] &\triangleq \lambda z_1 \ldots \lambda z_S.(z_i\ B_1 \ldots B_{\alpha_i}) \\
[\![\boldsymbol{\lambda} f_i(X_1, \ldots, X_p).A]\!] &\triangleq \lambda y.(y\ \underbrace{x_\perp \ldots x_\perp}_{(\alpha_i - p)}\ \underbrace{x_\perp \ldots x_\perp}_{(i-1)}\ \lambda X_1 \ldots \lambda X_p.[\![A]\!]\ \ \underbrace{x_\perp \ldots x_\perp}_{(S-i)})
\end{aligned}
$$

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[\![\boldsymbol{\lambda} f_i(X_1, \ldots, X_p).A]\!] &\triangleq \lambda y.(y\ \underbrace{x_\perp \ldots x_\perp}_{(\alpha_i - p)}\ \underbrace{x_\perp \ldots x_\perp}_{(i-1)}\ \lambda X_1 \ldots \lambda X_p.[\![A]\!]\ \ \underbrace{x_\perp \ldots x_\perp}_{(S-i)}) \\
[\![\boldsymbol{\lambda} X.A]\!] &\triangleq \lambda X.[\![A]\!] \\
[\![A \bullet B]\!] &\triangleq [\![A]\!][\![B]\!]
\end{aligned}
$$

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[\![f_i]\!] &\triangleq \lambda x_1 \ldots \lambda x_{\alpha_i}.\ (\lambda z_1 \ldots \lambda z_S.(z_i\ x_1 \ldots x_{\alpha_i})) \\
[\![f_i \bullet B_1 \ldots \bullet B_{\alpha_i}]\!] &\triangleq \lambda z_1 \ldots \lambda z_S.(z_i\ B_1 \ldots B_{\alpha_i}) \\
[\![\boldsymbol{\lambda} f_i(X_1, \ldots, X_p).A]\!] &\triangleq \lambda y.(y\ \underbrace{x_\perp \ldots x_\perp}_{(\alpha_i - p)}\ \underbrace{x_\perp \ldots x_\perp}_{(i-1)}\ \lambda X_1 \ldots \lambda X_p.[\![A]\!]\ \ \underbrace{x_\perp \ldots x_\perp}_{(S-i)}) \\
[\![\boldsymbol{\lambda} X.A]\!] &\triangleq \lambda X.[\![A]\!] \\
[\![A \bullet B]\!] &\triangleq [\![A]\!][\![B]\!] \\
[\![A; B]\!] &\triangleq \lambda x_1 \ldots \lambda x_\alpha.\ \Big(\big(\lambda z.([\![A]\!] x_1 \ldots x_\alpha)\big)([\![B]\!] x_1 \ldots x_\alpha) \Big)
\end{aligned}
$$

Examples

$$(Success) \qquad \underbrace{\Big(\lambda y.(y(\lambda X.X)x_{\perp})\Big)}_{[\![\boldsymbol{\lambda} f(X).X]\!]}\underbrace{\lambda z_1 \lambda z_2.(z_1 Y)}_{[\![f^{\bullet} Y]\!]}$$

Examples

$$(Success) \qquad \underbrace{\Big(\lambda y.(y(\lambda X.X)x_{\perp})\Big)}_{[\![\boldsymbol{\lambda} f(X).X]\!]}\underbrace{\lambda z_1\lambda z_2.(z_1 Y)}_{[\![f^{\bullet} Y]\!]}$$

$$\mapsto_\beta \quad \underbrace{\Big(\lambda z_1\lambda z_2.(z_1 Y)\Big)(\lambda X.X)x_{\perp}}_{[\![[f(X)\ll f(Y)]X]\!]}$$

Examples

$$(Success) \qquad \underbrace{\Big(\lambda y.(y(\lambda X.X)x_\perp)\Big)}_{[\![\boldsymbol{\lambda} f(X).X]\!]}\underbrace{\lambda z_1 \lambda z_2.(z_1 Y)}_{[\![f^\bullet Y]\!]}$$

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$$\mapsto_\beta \quad \Big(\lambda z_2.((\lambda X.X)Y)\Big) x_\perp$$

Examples

$$(Success) \qquad \underbrace{\Big(\lambda y.(y(\lambda X.X)x_\perp)\Big)}_{[\![\boldsymbol{\lambda} f(X).X]\!]}\underbrace{\lambda z_1 \lambda z_2.(z_1 Y)}_{[\![f^\bullet Y]\!]}$$

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$$\mapsto_\beta \quad (\lambda X.X)Y$$

Examples

$$(Success) \qquad \underbrace{\Big(\lambda y.(y(\lambda X.X)x_{\perp})\Big)}_{[\![\boldsymbol{\lambda} f(X).X]\!]}\underbrace{\lambda z_1\lambda z_2.(z_1 Y)}_{[\![f^{\bullet}Y]\!]}$$

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Examples

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$$\mapsto_\beta \quad \big(\lambda z_2.((\lambda X.X)Y)\big)x_\perp$$

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$$(Failure) \quad \underbrace{\big(\lambda y.(y(\lambda X.X)x_\perp)\big)}_{[\![\boldsymbol{\lambda} f(X).X]\!]} \underbrace{\lambda z_1 \lambda z_2.(z_2 Y)}_{[\![g^\bullet Y]\!]}$$

Examples

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Examples

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$$\mapsto_\beta \quad \big(\lambda z_2.(z_2 Y)\big)x_\perp$$

$$\mapsto_\beta \quad x_\perp Y$$

The type of a constant

$$\vdash [\![f_1]\!] = \lambda x_1.\lambda z_1.\lambda z_2.(z_1\ x_1) : \sigma_1 \to (\sigma_1 \to \beta) \to (\sigma_1 \to \beta) \to \beta$$

$$\vdash [\![f_1(B_1)]\!] : (\sigma_1 \to \beta) \to (\sigma_1 \to \beta) \to \beta$$

$$\vdash [\![\boldsymbol{\lambda} f_1(X_1).A]\!] = \lambda y.\left(y\ \left(\lambda X_1.[\![A]\!]\right)\ x_\perp\right) : ((\sigma_1 \to \tau) \to (\sigma_1 \to \tau) \to \gamma) \to \gamma$$

where $x_\perp : \sigma_1 \to \tau$

$(\sigma_1 \to \tau) \to (\sigma_1 \to \tau) \to \gamma \ = \ (\sigma_1 \to \beta) \to (\sigma_1 \to \beta) \to \beta$ thus $\tau = \beta = \gamma$

Enhanced translation

$$\vdash x_\perp \;:\; \perp \triangleq \Pi(\beta:*).\beta$$

$$\{\sigma_1,\ldots,\sigma_\alpha\} \triangleq \Pi(\beta:*).\big([\sigma_1 \to \ldots \sigma_\alpha \to \beta]^S \to \beta\big)$$

$$\text{where } [\sigma]^S \to \tau \triangleq \underbrace{\sigma \to \ldots \to \sigma}_{S} \to\!\triangleright\, \tau$$

$$[\![f_i]\!] \triangleq \lambda x_1.\lambda x_2.\lambda(\beta:*)\;(\lambda z_1 \ldots \lambda z_S.(z_i x_1 x_2)) : \sigma_1 \to \sigma_2 \to \{\sigma_1,\sigma_2\}$$

$$[\![\boldsymbol{\lambda} f_i(X_1,X_2).A]\!] \triangleq \lambda y.\big(y\;\tau\;\overrightarrow{(x_\perp \tau_0)}\;\;\lambda X_1.\lambda X_2.[\![A]\!]\;\overrightarrow{(x_\perp \tau_0)}\big) : \{\sigma_1,\sigma_2\} \to \tau$$

where $[\![\Gamma]\!] \vdash_{\mathsf{F}\omega} [\![A]\!] : \tau$

and $[\![\Gamma]\!] \vdash_{\mathsf{F}\omega} \lambda X_1.\lambda X_2.[\![A]\!] : \tau_0$

The type of a variable

$$
\begin{array}{rll}
(\Xi : *), (X : Y \rightharpoondown_{Y:\Xi} \Xi) & \vdash_\rho X & : \mathbf{\Pi}(Y{:}\Xi) \,.\, \Xi \\[1em]
(\Xi : *) & \vdash_\rho \boldsymbol{\lambda}(Y{:}\Xi) \,.\, Y & : \mathbf{\Pi}(Y{:}\Xi) \,.\, \Xi \\
(\Xi : *), (a : \Xi) & \vdash_\rho \boldsymbol{\lambda}(Y{:}\Xi) \,.\, a & : \mathbf{\Pi}(Y{:}\Xi) \,.\, \Xi \\
(\Xi : *), (f : Y \rightharpoondown_{Y:\Xi} \Xi) & \vdash_\rho f & : \mathbf{\Pi}(Y{:}\Xi) \,.\, \Xi
\end{array}
$$

The type of a variable

$$
\begin{array}{rll}
(\Xi : *), (X : Y \rightharpoonup_{Y:\Xi} \Xi) & \vdash_\rho X & : \mathbf{\Pi}(Y{:}\Xi) \,.\, \Xi \\
\\
(\Xi : *) & \vdash_\rho \boldsymbol{\lambda}(Y{:}\Xi) \,.\, Y & : \mathbf{\Pi}(Y{:}\Xi) \,.\, \Xi \\
(\Xi : *), (a : \Xi) & \vdash_\rho \boldsymbol{\lambda}(Y{:}\Xi) \,.\, a & : \mathbf{\Pi}(Y{:}\Xi) \,.\, \Xi \\
(\Xi : *), (f : Y \rightharpoonup_{Y:\Xi} \Xi) & \vdash_\rho f & : \mathbf{\Pi}(Y{:}\Xi) \,.\, \Xi
\end{array}
$$

$$
\begin{array}{ll}
\Gamma \vdash_{\mathsf{F}\omega} \lambda(\beta_Y : *).\lambda(Y : \beta_Y).Y & : \Pi(\beta_Y : *).\big(\beta_Y \to \beta_Y\big) \\
\Gamma \vdash_{\mathsf{F}\omega} \lambda(\beta_Y : *).\lambda(Y : \beta_Y).[\![a]\!] & : \Pi(\beta_Y : *).\big(\beta_Y \to \{\emptyset\}\big) \\
\Gamma \vdash_{\mathsf{F}\omega} [\![f]\!] & : \Pi(\beta_Y : *).\big(\beta_Y \to \{\beta_Y\}\big)
\end{array}
$$

Use of types depending on types

$$(\Xi : *) \quad \vdash_\rho \quad X \quad : \quad \mathbf{\Pi}(Y{:}\Xi)\,.\,\Xi$$

$$\beta_X : * \to * \quad \vdash_{\mathsf{F}\omega} \quad [\![X]\!] \quad : \quad \Pi(\beta_Y : *).\ \ (\beta_Y \to \beta_X \beta_Y)$$

$[\![\boldsymbol{\lambda} Y\ .\ Y]\!]$	$\Pi(\beta_Y : *).\beta_Y \to \beta_Y$	$\beta_X := \lambda(\beta : *).\beta$
$[\![\boldsymbol{\lambda} Y\ .\ a]\!]$	$\Pi(\beta_Y : *).\beta_Y \to \{\emptyset\}$	$\beta_X := \lambda(\beta : *).\{\emptyset\}$
$[\![f]\!]$	$\Pi(\beta_Y : *).\beta_Y \to \{\beta_Y\}$	$\beta_X := \lambda(\beta : *).\{\beta\}$

Steps of the proof

- Faithful reductions

- Typability of the translated terms

- Strong normalization

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  If $A \mapsto_{\rho\sigma\delta} B$, then $[\![A]\!] \overset{\beta}{\mapsto\!\!\!\to} [\![B]\!]$ in at least one step.

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  Giving a fresh variable Z,

$$\Sigma, \Gamma \vdash_{\rho} A : \Phi \quad \Rightarrow \quad \exists \tau_A, \quad [\![\Gamma]\!] \vdash_{\mathsf{F}\omega} [\![A]\!] : [\![\Phi]\!]^Z_\emptyset [\beta_Z := \tau_A]$$

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  If $A \mapsto_{\rho\sigma\delta} B$, then $[\![A]\!] \overset{\beta}{\mapsto\!\!\!\twoheadrightarrow} [\![B]\!]$ in at least one step.

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- Strong normalization

  If $\Sigma, \Gamma \vdash_\rho A : \Phi$ then A and Φ are strongly normalizing.

Towards an extension of Curry-Howard

- Some open questions
  - ➩ What do the types $\Pi A{:}\Delta\ .\ C$ and $[A \ll B].C$ mean ?
  - ➩ Are the structures suitable for products or sums ?

Towards an extension of Curry-Howard

- Some open questions

  ➪ What do the types $\Pi A{:}\Delta \, . \, C$ and $[A \ll B].C$ mean ?

  ➪ Are the structures suitable for products or sums ?

- Some partial answers

  ➪ The λ-term $[\![f(X_1, \ldots , X_n)]\!]$ has type $\bigwedge_{i=1..n} X_i$.

  ➪ Proof assistant based on the ρ-calculus: deduction and computation at the same level.

  ➪ Use of the typed fixpoints to model trustworthy computations steps.

Bonus tracks

Example with a delayed constraint

$$(\boldsymbol{\lambda} Y.\boldsymbol{\lambda}(f \bullet X.X) \bullet Y) \bullet (f \bullet a)$$

Example with a delayed constraint

$$
\begin{aligned}
& (\boldsymbol{\lambda} Y.\boldsymbol{\lambda}(f\bullet X.X)\bullet Y)\bullet(f\bullet a) \\
\mapsto_\rho \quad & (\boldsymbol{\lambda} Y.[f\bullet X \ll Y]X)\bullet(f\bullet a)
\end{aligned}
$$

Example with a delayed constraint

$$
\begin{array}{ll}
 & (\boldsymbol{\lambda} Y.\boldsymbol{\lambda}(f\bullet X.X)\bullet Y)\bullet(f\bullet a) \\
\mapsto_\rho & (\boldsymbol{\lambda} Y.[f\bullet X \ll Y]X)\bullet(f\bullet a) \\
\mapsto_\rho & [Y \ll f\bullet a][f\bullet X \ll Y]X
\end{array}
$$

Example with a delayed constraint

$$
\begin{array}{ll}
 & (\boldsymbol{\lambda} Y.\boldsymbol{\lambda}(f\bullet X.X)\bullet Y)\bullet(f\bullet a) \\
\mapsto_\rho & (\boldsymbol{\lambda} Y.[f\bullet X \ll Y]X)\bullet(f\bullet a) \\
\mapsto_\rho & [Y \ll f\bullet a][f\bullet X \ll Y]X \\
\mapsto_\sigma & [f\bullet X \ll f\bullet a]X
\end{array}
$$

Example with a delayed constraint

$$
\begin{array}{ll}
 & (\boldsymbol{\lambda} Y.\boldsymbol{\lambda}(f{\bullet}X.X){\bullet}Y){\bullet}(f{\bullet}a) \\
\mapsto_\rho & (\boldsymbol{\lambda} Y.[f{\bullet}X \ll Y]X){\bullet}(f{\bullet}a) \\
\mapsto_\rho & [Y \ll f{\bullet}a][f{\bullet}X \ll Y]X \\
\mapsto_\sigma & [f{\bullet}X \ll f{\bullet}a]X \\
\mapsto_\sigma & a
\end{array}
$$

Example with a delayed constraint (cont'd)

$$(\lambda Y.\Big(\underbrace{(\lambda y.(y x_{\perp}(\lambda X.X)))}_{[\![\boldsymbol{\lambda} f^{\bullet} X.X]\!]} Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

$$
\begin{array}{ll}
 & (\lambda Y.\Big(\underbrace{(\lambda y.(y x_\perp(\lambda X.X)))}_{[\![\boldsymbol{\lambda} f^\bullet X.X]\!]}\,Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]}\,\underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big) \\
\stackrel{\beta}{\longmapsto} & \underbrace{(\lambda Y.(Y x_\perp(\lambda X.X)))}_{[\![\boldsymbol{\lambda} Y.[f^\bullet X \ll Y]X]\!]}\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]}\,\underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big)
\end{array}
$$

$$(\lambda Y.\Big(\underbrace{(\lambda y.(y x_\perp (\lambda X.X)))}_{[\![\boldsymbol{\lambda} f^\bullet X.X]\!]} Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]} \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

$$\stackrel{\beta}{\longmapsto} \quad \underbrace{\big(\lambda Y.(Y x_\perp (\lambda X.X))\big)}_{[\![\boldsymbol{\lambda} Y.[f^\bullet X \ll Y] X]\!]} \Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]} \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

$$\stackrel{\beta}{\longmapsto} \quad \big(\lambda Y.(Y x_\perp (\lambda X.X))\big)\big(\lambda z_1 \lambda z_2.(z_2 \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]})\big)$$

$$
\begin{aligned}
& (\lambda Y.\Big(\underbrace{(\lambda y.(y x_{\perp}(\lambda X.X)))}_{[\![\boldsymbol{\lambda} f^{\bullet} X.X]\!]} Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big) \\
\overset{\beta}{\longmapsto}\quad & \underbrace{\big(\lambda Y.(Y x_{\perp}(\lambda X.X))\big)}_{[\![\boldsymbol{\lambda} Y.[f^{\bullet} X \ll Y] X]\!]}\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big) \\
\overset{\beta}{\longmapsto}\quad & \big(\lambda Y.(Y x_{\perp}(\lambda X.X))\big)\big(\lambda z_1 \lambda z_2.(z_2 \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]})\big) \\
\overset{\beta}{\longmapsto}\quad & \big(\lambda z_1 \lambda z_2.(z_2(\lambda u_1 \lambda u_2.u_1))\big) x_{\perp}(\lambda X.X) \qquad\qquad (f \neq a)
\end{aligned}
$$

$$
\begin{array}{rl}
& (\lambda Y.\Big(\underbrace{(\lambda y.(y x_{\perp}(\lambda X.X)))}_{[\![\boldsymbol{\lambda} f^{\bullet} X.X]\!]} Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]} \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big) \\
\stackrel{\beta}{\longmapsto} & \underbrace{\big(\lambda Y.(Y x_{\perp}(\lambda X.X))\big)}_{[\![\boldsymbol{\lambda} Y.[f^{\bullet} X \ll Y] X]\!]} \Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]} \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big) \\
\stackrel{\beta}{\longmapsto} & \big(\lambda Y.(Y x_{\perp}(\lambda X.X))\big)\big(\lambda z_1 \lambda z_2.(z_2 \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]})\big) \\
\stackrel{\beta}{\longmapsto} & \big(\lambda z_1 \lambda z_2.(z_2(\lambda u_1 \lambda u_2.u_1))\big) x_{\perp}(\lambda X.X) \qquad (f \neq a) \\
\stackrel{\beta}{\longmapsto} & \big(\lambda z_2.(z_2(\lambda u_1 \lambda u_2.u_1))\big)(\lambda X.X) \qquad (f = f)
\end{array}
$$

$$(\lambda Y.\big(\underbrace{(\lambda y.(y x_\perp(\lambda X.X)))}_{[\![\boldsymbol{\lambda} f^\bullet X.X]\!]} Y\big))\big(\underbrace{(\lambda x_1.\lambda z_1\lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]}\big)$$

$$\stackrel{\beta}{\longmapsto} \quad \underbrace{\big(\lambda Y.(Y x_\perp(\lambda X.X))\big)}_{[\![\boldsymbol{\lambda} Y.[f^\bullet X \ll Y] X]\!]}\big(\underbrace{(\lambda x_1.\lambda z_1\lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]}\big)$$

$$\stackrel{\beta}{\longmapsto} \quad \big(\lambda Y.(Y x_\perp(\lambda X.X))\big)\big(\lambda z_1\lambda z_2.(z_2\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]})\big)$$

$$\stackrel{\beta}{\longmapsto} \quad \big(\lambda z_1\lambda z_2.(z_2(\lambda u_1\lambda u_2.u_1))\big)x_\perp(\lambda X.X) \qquad (f \neq a)$$

$$\stackrel{\beta}{\longmapsto} \quad \big(\lambda z_2.(z_2(\lambda u_1\lambda u_2.u_1))\big)(\lambda X.X) \qquad (f = f)$$

$$\stackrel{\beta}{\longmapsto} \quad (\lambda X.X)(\lambda u_1\lambda u_2.u_1)$$

$$(\lambda Y.\big(\underbrace{(\lambda y.(y x_\perp(\lambda X.X)))}_{[\![\boldsymbol{\lambda} f^\bullet X.X]\!]}\,Y\big))\Big(\underbrace{(\lambda x_1.\lambda z_1\lambda z_2.(z_2 x_1))}_{[\![f]\!]}\,\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

$$\overset{\beta}{\longmapsto}\quad \underbrace{(\lambda Y.(Y x_\perp(\lambda X.X)))}_{[\![\boldsymbol{\lambda} Y.[f^\bullet X \ll Y]X]\!]}\Big(\underbrace{(\lambda x_1.\lambda z_1\lambda z_2.(z_2 x_1))}_{[\![f]\!]}\,\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

$$\overset{\beta}{\longmapsto}\quad \big(\lambda Y.(Y x_\perp(\lambda X.X))\big)\big(\lambda z_1\lambda z_2.(z_2\,\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]})\big)$$

$$\overset{\beta}{\longmapsto}\quad \big(\lambda z_1\lambda z_2.(z_2(\lambda u_1\lambda u_2.u_1))\big)x_\perp(\lambda X.X) \qquad (f \neq a)$$

$$\overset{\beta}{\longmapsto}\quad \big(\lambda z_2.(z_2(\lambda u_1\lambda u_2.u_1))\big)(\lambda X.X) \qquad (f = f)$$

$$\overset{\beta}{\longmapsto}\quad (\lambda X.X)(\lambda u_1\lambda u_2.u_1)$$

$$\overset{\beta}{\longmapsto}\quad (\lambda u_1\lambda u_2.u_1)$$

$$=\quad [\![a]\!]$$

"Untyped" Matching theories I

Empty theory $\mathbb{T}_{\emptyset}$ of equality (up to α-conversion)

$$\frac{\Vdash T_1 = T_2 \quad \Vdash T_2 = T_3}{\Vdash T_1 = T_3}\ (Trans) \qquad \frac{\Vdash T_1 = T_2}{\Vdash T_2 = T_1}\ (Symm)$$

$$\frac{\Vdash T_1 = T_2}{\Vdash T_{3\lceil T_1 \rceil_p} = T_{3\lceil T_2 \rceil_p}}\ (Cont) \qquad \frac{}{\Vdash t = t}\ (Refl)$$

$T_{1\lceil T_2 \rceil_p}$: term T_1 with term T_2 at position p

BACK

“Untyped” Matching theories II

Theory of Associativity $\mathbb{T}_{A(f)}$ (resp. Commutativity $\mathbb{T}_{C(f)}$) is defined as $\mathbb{T}_{\emptyset}$ plus:

$$\frac{}{\Vdash f(f(T_1, T_2), T_3) = f(T_1, f(T_2, T_3))}\ (Assoc)$$

$$\frac{}{\Vdash f(T_1, T_2) = f(T_2, T_1)}\ (Comm)$$

BACK

“Untyped” Matching theories III

Theory of Idempotency $\mathbb{T}_{I(f)}$ is defined as $\mathbb{T}_{\emptyset}$ plus the axiom

$$\frac{}{\Vdash f(T,T) = t}\ (Idem)$$

Theory of Neutral Element $\mathbb{T}_{N(f^0)}$ is defined as $\mathbb{T}_{\emptyset}$ plus

$$\frac{}{\Vdash f(0,T) = T}\ (0\text{--}Left) \qquad \frac{}{\Vdash f(T,0) = T}\ (0\text{--}Right)$$

BACK

"Untyped" Matching theories IV

The Theory of Stuck $\mathbb{T}_{\mathsf{stk}}$ is defined as $\mathbb{T}_{N(,\mathsf{stk})}$ plus the axioms

$$\frac{\forall\theta_1,\theta_2,\ \forall C,\ A\theta_2 \mapsto\!\!\!\!\rightarrow_{\rho\sigma\delta} C \ \Rightarrow\ C \not\equiv P\theta_1}{\Vdash [P \ll A]B = \mathsf{stk}}$$

$$\frac{}{\Vdash \mathsf{stk}\bullet T = \mathsf{stk}}$$

Examples

$\Vdash [3 \ll 4]5 = \mathsf{stk}$

$\nVdash [3 \ll 3]5 = \mathsf{stk}$ $\qquad$ $\nVdash [3 \ll \mathcal{X}]5 = \mathsf{stk}$

BACK

"Untyped" Matching theories V

Theory of the Lambda Calculus of Objects $\mathbb{T}_{\lambda\mathcal{O}bj}$ is obtained by considering the symbol ";" as associative and stk as its neutral element, $i.e.$:

$$\mathbb{T}_{\lambda\mathcal{O}bj} = \mathbb{T}_{A(;)} \cup \mathbb{T}_{\mathsf{stk}}$$

Theory of the Object Calculus $\mathbb{T}_{\varsigma\mathcal{O}bj}$ is obtained by considering the symbol ";" as associative and commutative and stk as its neutral element, $i.e.$:

$$\mathbb{T}_{\varsigma\mathcal{O}bj} = \mathbb{T}_{A(;)} \cup \mathbb{T}_{C(;)} \cup \mathbb{T}_{\mathsf{stk}} = \mathbb{T}_{\lambda\mathcal{O}bj} \cup \mathbb{T}_{C(;)}$$

THEORIES RECORDS

The Matching Algorithm for $\mathbb{T}_\emptyset$

The matching substitution solving a matching equation can be computed by the following *matching reduction system*:

$$(Appl) \qquad (T_1 \bullet T_2) \lll (T_3 \bullet T_4) \rightsquigarrow T_1 \lll T_3 \wedge T_2 \lll T_4$$

$$(Struct) \quad (T_1 ; T_2) \lll (T_3 ; T_4) \rightsquigarrow T_1 \lll T_3 \wedge T_2 \lll T_4$$

Example

$f(\mathcal{X}, \mathcal{Y}) \lll f(a, b) \rightsquigarrow \mathcal{X} \lll a \wedge \mathcal{Y} \lll b$ successful

$f(\mathcal{X}, \mathcal{X}) \lll f(a, b) \rightsquigarrow \mathcal{X} \lll a \wedge \mathcal{Y} \lll b$ unsuccessful

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On the (non-)confluence

Variables in applicative position

$$(\boldsymbol{\lambda}\mathcal{X}{\bullet}\mathcal{Y}.\mathcal{X})((\boldsymbol{\lambda}\mathcal{Z}.\mathcal{Z})a)$$

$$\swarrow_{\rho\sigma\delta} \qquad\qquad \searrow_{\rho\sigma\delta}$$

$$[\mathcal{X}{\bullet}\mathcal{Y} \ll (\boldsymbol{\lambda}\mathcal{Z}.\mathcal{Z})a]\mathcal{X} \qquad\qquad (\boldsymbol{\lambda}\mathcal{X}{\bullet}\mathcal{Y}.\mathcal{X})([\mathcal{Z} \ll a]\mathcal{Z})$$

$$\Downarrow_{\rho\sigma\delta} \qquad\qquad \Downarrow_{\rho\sigma\delta}$$

$$\boldsymbol{\lambda}\mathcal{Z}.\mathcal{Z} \qquad\qquad (\boldsymbol{\lambda}\mathcal{X}{\bullet}\mathcal{Y}.\mathcal{X})a$$

On the (non-)confluence

Variables in applicative position

$$(\boldsymbol{\lambda}\mathcal{X}\bullet\mathcal{Y}.\mathcal{X})((\boldsymbol{\lambda}\mathcal{Z}.\mathcal{Z})a)$$

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$$\Downarrow_{\rho\sigma\delta} \qquad\qquad \Downarrow_{\rho\sigma\delta}$$

$$\boldsymbol{\lambda}\mathcal{Z}.\mathcal{Z} \qquad\qquad (\boldsymbol{\lambda}\mathcal{X}\bullet\mathcal{Y}.\mathcal{X})a$$

Rigid Pattern Condition (RPC) [van Oostrom 90]

$\mathcal{P} \triangleq \{T \in NF(\rho\sigma\delta) \mid T$ is "linear" with no "active" variables$\}$

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