Closed Reduction and Director Strings

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- Well, because this is our playground (*eg.* reuse compiler).

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- In Coq, terms are strongly normalisable, so why should we abide to the *"do not ever reduce under lambdas"* credo ?
- Well, because this is our playground (*eg.* reuse compiler).
- Let's instead take a fresh start and think first about an efficient **strategy**

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- Hence the name of **closed** reduction

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  - full simulation of β is *not* required

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- λ-calculus

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- and *"explicit resource management"* (inspired by linear logic)

$$
\begin{aligned}
t, u \ ::= \ & x \mid \lambda x.t \mid t\ u \\
& \mid t[x/u] \\
& \mid \epsilon_x.t \\
& \mid \delta_x^{y,z}.t
\end{aligned}
$$

and we impose linearity

Closed reduction

Name		Reduction	Condition
Beta	$(\lambda x.t)u$	$\to t[x/u]$	
Var_1	$x[x/v]$	$\to v$	
Var_2	$y[x/v]$	$\to y$	$x \neq y$
App	$(t\,u)[x/v]$	$\to (t[x/v])(u[x/v])$	
Lam_1	$(\lambda x.t)[x/v]$	$\to (\lambda x.t)$	
Lam_2	$(\lambda y.t)[x/v]$	$\to \lambda y.t[x/v]$	$x \notin \mathsf{fv}(t) \vee y \notin \mathsf{fv}(v)$
Lam_3	$(\lambda y.t)[x/v]$	$\to \lambda z.t[z/y][x/v]$	$x \in \mathsf{fv}(t), y \in \mathsf{fv}(v), z$ fresh
Comp	$t[y/w][x/v]$	$\to t[x/v][y/w[x/v]]$	$y \notin \mathsf{fv}(v)$

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where $x, y \in \mathrm{fv}(t)$

$\Rightarrow$ $\mathrm{fv}(\lambda x.t) = \emptyset$ in Beta cuts the left branch

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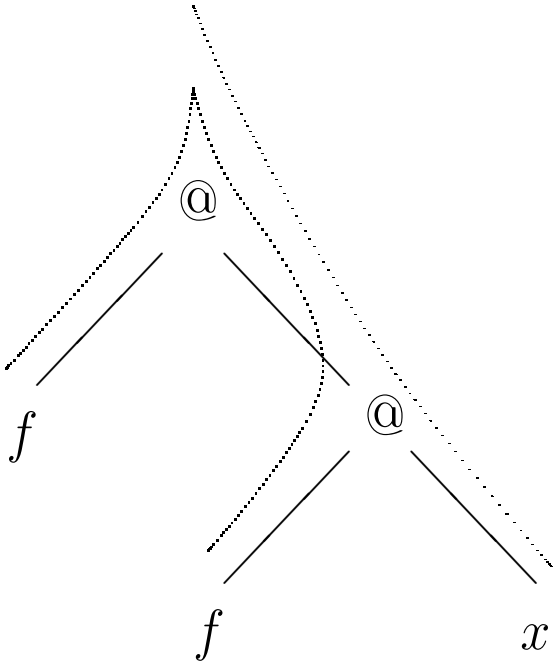
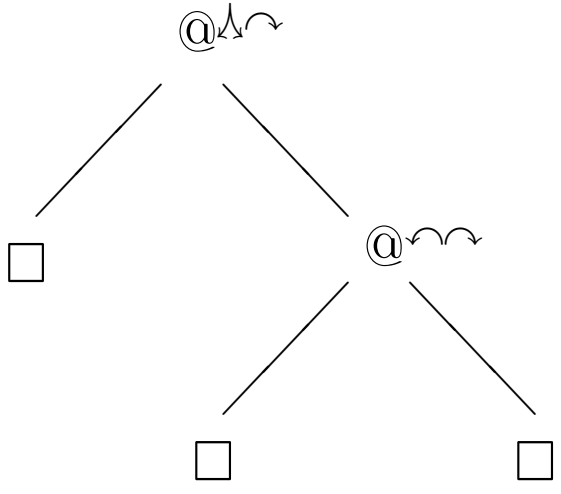
- an old idea: Kennaway and Sleep (for combinators)

An example

$(f\ (f\ x))[f/F][x/X]$



$(\square\ (\square\ \square)\curvearrowright\curvearrowright)\curvearrowleft\curvearrowright$



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- **terms** have to satisfy well-formedness conditions

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- rules are **simpler** for closed reduction (note that we associate erasing with abstraction as λ^- and copy with application thanks to $\curlywedge$)

Closed reduction with director strings

Name			
Beta	$((\lambda t)^{\epsilon}\ u)\curvearrowright^{n}$	$\longrightarrow$	$(t[u])\curvearrowright^{n}$
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- adequate for reduction of **closed** terms to **whnf**

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  - well-known trick: successively **freeze** free variables

Experimental results

term	CRAM	CBV	BOHM
2 2 I I	61 (9)	82 (11)	40 (9)
2 2 2 I I	140 (19)	302 (42)	100 (16)
3 2 2 I I	248 (33)	3508 (531)	184 (21)
5 5 I I	217 (33)	26669 (3913)	229 (33)
5 2 2 I I	832 (109)	–	847 (31)
2 2 2 2 2 I I	1507714 (196655)	–	1074037060 (61)

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- techniques of **compilation**, *etc.* still to be investigated (non-trivial work)
- seems well-suited for **proof assistants** (strongly normalising terms, higher-order functions, impredicativity)
- for **functional languages**, should be associated with a weaker strategy either in an *ad hoc* way (*cf.* if␣then␣else in ML) or with *optimistic evaluation*

That's all, folks

Questions ?