Characterizing strong normalization in a language with control operators

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Plan of the talk

- The calculus $\overline{\lambda}\mu\widetilde{\mu}$
  - Implicative sequent calculus,
  - Curien-Herbelin's calculus $\overline{\lambda}\mu\widetilde{\mu}$,
  - Call by name and call by value,
- Characterizing strongly normalizable terms
  - Definite systems,
  - The system $\mathcal{M}^{\cap\cup}$,
  - Subject reduction,
  - The strong reduction theorem.

THE CALCULUS $\overline{\lambda}\mu\widetilde{\mu}$

Sequent calculus

A model of deduction

The implicative sequent calculus

Propositions are made only

- of propositional variables
- and of the implication connector.

The implicative sequent calculus (the rules)

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (ax)$$

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$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

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$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ (cut)$$

The active proposition

The active proposition is the proposition on the lower part of a rule which is "split" by the rule.

For instance in

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} (\to L)$$

the active proposition is $A \to B$.

The active proposition

It makes sense to track the active proposition and to suppose that A and B become the new active propositions:

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \, (\to L)$$

Similarly

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \, (\to R)$$

We have to prove B using the proposition A
and to split B if necessary.

The rules of the implicative sequent calculus with active propositions

$$\frac{}{\Gamma, A \vdash \Delta, A} \; (L - ax) \qquad \frac{}{\Gamma, A \vdash \Delta, A} \; (R - ax)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \; (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \; (\to R)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta} \; (cut)$$

The rules of the implicative sequent calculus with active propositions

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (L-ax) \qquad \frac{}{\Gamma, A \vdash \Delta, A}\ (R-ax)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\ (\rightarrow L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\ (\rightarrow R)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ (cut)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A, \Delta}\ (\mu) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ (\widetilde{\mu})$$

A proof of Peirce law

$$
\dfrac{
\dfrac{
(A \to B) \to A \vdash (A \to B) \to A, A
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
(A \to B) \to A, A \vdash A, B, A \qquad (A \to B) \to A, A, A \vdash A, B
}{
(A \to B) \to A, A \vdash B, A
}\,(cut)
}{
(A \to B) \to A, A \vdash B, A
}\,(\mu)
}{
(A \to B) \to A \vdash A \to B, A
}\,(\to R)
\qquad
(A \to B) \to A, A \vdash A
}{
(A \to B) \to A, (A \to B) \to A \vdash A
}\,(\to L)
}{
\dfrac{
(A \to B) \to A \vdash A
}{
(A \to B) \to A \vdash A
}\,(\mu)
}\,(cut)
}{
\vdash ((A \to B) \to A) \to A
}\,(\to R)
$$

The model of computation: the Curien-Herbelin calculus

$$\overline{\lambda}\mu\widetilde{\mu}$$

A model of computation

$\overline{\lambda}\mu\widetilde{\mu}$ relies on capsules $\langle r \parallel e \rangle$ that contain two constituents:

- a caller r
- and a callee e .

with the syntax

$$
\begin{array}{lcl}
c & ::= & \langle r \parallel e \rangle \\
r & ::= & x \mid \lambda x.r \mid \mu\alpha.c \\
e & ::= & \alpha \mid r \bullet e \mid \widetilde{\mu}x.c
\end{array}
$$

The reductions

$$
\begin{array}{lrcl}
(\lambda) & \langle \lambda x \cdot r \parallel r' \bullet e \rangle & \longrightarrow & \langle r[x \leftarrow r'] \parallel e \rangle \\
(\mu) & \langle \mu\alpha \cdot c \parallel e \rangle & \longrightarrow & c[\alpha \leftarrow e] \\
(\widetilde{\mu}) & \langle r \parallel \widetilde{\mu} x \cdot c \rangle & \longrightarrow & c[x \leftarrow r]
\end{array}
$$

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(\widetilde{\mu}) & \langle r \parallel \widetilde{\mu}x \cdot c \rangle & \longrightarrow & c[x \leftarrow r]
\end{array}
$$

The system is ambiguous !

$$\langle \mu\alpha.c \parallel \widetilde{\mu}x.c' \rangle$$

$$c[\alpha \leftarrow \widetilde{\mu}x.c'] \qquad\qquad c'[x \leftarrow \mu\alpha \cdot c]$$

The correspondence between the sequent calculus and Curien-Herbelin's calculus

The type judgments

Thanks to colors, I will consider three types of judgments

They can be seen as annotations of sequent calculus judgments.

Judgments for capsules

$$c : x_1 : A_1, \ldots, x_p : A_p \vdash \alpha_1 : B_1, \ldots, \alpha_q : B_q$$

or in short $c : \Gamma \vdash \Delta$,

Judgments for capsules

In $c : x_1 : A_1, \ldots, x_p : A_p \vdash \alpha_1 : B_1, \ldots, \alpha_q : B_q$,

one says that

- c takes the x_i as arguments with type A_i
- c expects a continuation α_j with type B_j.

Judgments for callers

$$x_1 : A_1, \ldots, x_p : A_p \vdash r : A, \alpha_1 : B_1, \ldots, \alpha_q : B_q$$

or in short $\Gamma \vdash r : A, \Delta$,

or $\Gamma \vdash \boxed{r : A}, \Delta$, when one does not have color.

Judgments for callees

$$x_1 : A_1, \ldots, x_p : A_p, e : A \vdash \alpha_1 : B_1, \ldots, \alpha_q : B_q$$

or in short $\Gamma, e : A \vdash \Delta$.

or $\Gamma, \boxed{e : A} \vdash \Delta$, when one does not have color.

The type system $G^{\rightarrow}$

$$\frac{}{\Gamma, \alpha : A \vdash \alpha : A, \Delta}\ (L-ax) \qquad \frac{}{\Gamma, x : A \vdash x : A, \Delta}\ (R-ax)$$

$$\frac{\Gamma \vdash r : A, \Delta \qquad \Gamma, e : B \vdash \Delta}{\Gamma, r \bullet e : A \rightarrow B \vdash \Delta}\ (\rightarrow L) \qquad \frac{\Gamma, x : A \vdash r : B, \Delta}{\Gamma \vdash \lambda x.r : A \rightarrow B, \Delta}\ (\rightarrow R)$$

$$\frac{\Gamma \vdash r : A, \Delta \qquad \Gamma, e : A \vdash \Delta}{\langle r \parallel e \rangle : (\Gamma \vdash \Delta)}\ (cut)$$

$$\frac{c : (\Gamma \vdash \beta : B, \Delta)}{\Gamma \vdash \mu\beta.c : B, \Delta}\ (\mu) \qquad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma, \widetilde{\mu}x.c : A \vdash \Delta}\ (\widetilde{\mu})$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\,(L-ax) \qquad \frac{}{\Gamma, A \vdash \Delta, A}\,(R-ax)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,(\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\,(\to R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,(cut) \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash B, \Delta}\,(\mu) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,(\widetilde{\mu})$$

sequent calculus

$$\frac{}{\Gamma, \alpha : A \vdash \alpha : A, \Delta}\,(L-ax) \qquad \frac{}{\Gamma, x : A \vdash x : A, \Delta}\,(R-ax)$$

$$\frac{\Gamma \vdash r : A, \Delta \quad \Gamma, e : B \vdash \Delta}{\Gamma, r \bullet e : A \to B \vdash \Delta}\,(\to L) \qquad \frac{\Gamma, x : A \vdash r : B, \Delta}{\Gamma \vdash \lambda x.r : A \to B, \Delta}\,(\to R)$$

$$\frac{\Gamma \vdash r : A, \Delta \quad \Gamma, e : A \vdash \Delta}{\langle r \parallel e\rangle : (\Gamma \vdash \Delta)}\,(cut) \qquad \frac{c : (\Gamma \vdash \beta : B, \Delta)}{\Gamma \vdash \mu\beta.c : B, \Delta}\,(\mu) \qquad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma, \widetilde{\mu}x.c : A \vdash \Delta}\,(\widetilde{\mu})$$

$G^{\to}$

Peirce law again

Let T be $(A \to B) \to A$.

$$\dfrac{x : T, y : A \vdash y : A, \beta : B, \alpha : A \qquad x : T, y : A, \alpha : A \vdash \alpha : A, \beta : B}{\langle y \parallel \alpha \rangle : (x : T, y : A \vdash \beta : B, \alpha : A)}\ (cut)$$

$$\dfrac{\langle y \parallel \alpha \rangle : (x : T, y : A \vdash \beta : B, \alpha : A)}{x : T, y : A \vdash \mu\beta.\langle y \parallel \alpha \rangle : B, \alpha : A}\ (\mu)$$

$$\dfrac{x : T, y : A \vdash \mu\beta.\langle y \parallel \alpha \rangle : B, \alpha : A}{x : T \vdash \lambda y.\mu\beta.\langle y \parallel \alpha \rangle : A \to B, \alpha : A}\ (\to R)$$

$$\dfrac{x : T \vdash \lambda y.\mu\beta.\langle y \parallel \alpha \rangle : A \to B, \alpha : A \qquad x : T, \alpha : A \vdash \alpha : A}{x : T, (\lambda y.\mu\beta.\langle y \parallel \alpha \rangle) \bullet \alpha : T \vdash \alpha : A}\ (\to L)$$

$$\dfrac{x : T \vdash x : T, \alpha : A \qquad x : T, (\lambda y.\mu\beta.\langle y \parallel \alpha \rangle) \bullet \alpha : T \vdash \alpha : A}{\langle x \parallel (\lambda y.\mu\beta.\langle y \parallel \alpha \rangle) \bullet \alpha \rangle : (x : T \vdash \alpha : A)}\ (cut)$$

$$\dfrac{\langle x \parallel (\lambda y.\mu\beta.\langle y \parallel \alpha \rangle) \bullet \alpha \rangle : (x : T \vdash \alpha : A)}{x : T \vdash \mu\alpha.\langle x \parallel (\lambda y.\mu\beta.\langle y \parallel \alpha \rangle) \bullet \alpha \rangle : A,}\ (\mu)$$

$$\dfrac{x : T \vdash \mu\alpha.\langle x \parallel (\lambda y.\mu\beta.\langle y \parallel \alpha \rangle) \bullet \alpha \rangle : A,}{\vdash \lambda x.\mu\alpha.\langle x \parallel (\lambda y.\mu\beta.\langle y \parallel \alpha \rangle) \bullet \alpha \rangle : ((A \to B) \to A) \to A,}\ (\to L)$$

Peirce law again

The term with type the Peirce law is

$$\lambda x.\mu\alpha.\langle x \parallel (\lambda y.\mu\beta.\langle y \parallel \alpha\rangle) \bullet \alpha\rangle.$$

Call by name

Call by value

Call by name / call by value

One considers an extended calculus

$$
\begin{array}{llll}
(\lambda) & \langle \lambda x \cdot r \parallel r' \bullet e \rangle & \longrightarrow & \langle r[x \leftarrow r'] \parallel e \rangle \\
(\mu) & \langle \mu\alpha \cdot c \parallel e \rangle & \longrightarrow & c[\alpha \leftarrow e] \\
(\widetilde{\mu}) & \langle r \parallel \widetilde{\mu} x \cdot c \rangle & \longrightarrow & c[x \leftarrow r]
\end{array}
$$

Call by name is when one gives priority to $(\widetilde{\mu})$.

Call by value is when one gives priority to (μ).

Call by name / call by value

One considers an extended calculus

$$\begin{array}{llcl} (\lambda') & \langle \lambda x \cdot r \parallel r' \bullet e \rangle & \longrightarrow & \langle r' \parallel \widetilde{\mu} x . \langle r \parallel e \rangle \rangle \\ (\mu) & \langle \mu \alpha \cdot c \parallel e \rangle & \longrightarrow & c[\alpha \leftarrow e] \\ (\widetilde{\mu}) & \langle r \parallel \widetilde{\mu} x \cdot c \rangle & \longrightarrow & c[x \leftarrow r] \end{array}$$

Call by name is when one gives priority to $(\widetilde{\mu})$.

Call by value is when one gives priority to (μ).

Characterizing strongly normalizable terms

Characterization of strong normalization

Let us abandon the type system and retain the language.

We want to characterize strong normalization by a type system.

Characterization of strong normalization

Let us abandon the type system and retain the language.

We want to characterize strong normalization by a type system.

Hence a system with intersection, but also with union by symmetry.

Typing judgments for capsules, callers, and callees are the same as for simple types, except that unions and intersections are allowed in building types.

Definite systems

A first idea: forbid union types $\not{r} : \not{A} \not\cup \not{B}$ for callers
and intersection types $\not{e} : \not{A} \not\cap \not{B}$ for callees.

Immediately a failure: in the presence of μ any type which can be the type of a callee variable can be the type of a caller term (and similarly for callees).

$$\frac{c : \quad \Gamma \vdash \alpha : A \cup B, \Delta}{\Gamma, \mu.\alpha.c : A \cup B \vdash \Delta} \; (\mu)$$

Definite systems

A better idea: simply forbid typing judgment whose bases contain variable typing of the form

$x : A \cup B$

and $\alpha : A \cap B$.

$$\Gamma = \{x_1 : x : A_1 \cap B_1, ..., \cancel{x_i : A_i \cup B_i}, ...\}.$$

$$\Delta = \{\alpha_1 : x : A_1 \cup B_1, ..., \cancel{\alpha_i : A_i \cap B_i}, ...\}.$$

Definition

① A type A is $\cap$-definite if it is a type variable, an arrow-type or it is $A_1 \cap A_2$, with each A_i a $\cap$-definite type.

A type A is $\cup$-definite if it is a type variable, an arrow-type or it is $A_1 \cup A_2$, with each A_i a $\cup$-definite type.

② A basis Γ is $\cap$-definite, if in each binding $x : A$ in Γ, A is a $\cap$-definite type.

A basis Δ is $\cup$-definite, if in each binding $\alpha : A$ in Δ, A is a $\cup$-definite type.

③ A typing judgment $\Gamma \vdash \Delta$ is definite if

- its typing base Γ is $\cap$- definite
- and its typing base Δ is $\cup$-definite.

The definite system

In each rule below the type bases are assumed definite.

$$\frac{\Gamma, x : A \ \vdash \ \Delta}{\Gamma, x : A \cap B \ \vdash \ \Delta} \ (\cap_{L-Var})$$

$$\frac{\Gamma, \ \ e : A \ \ \vdash \ \ \Delta}{\Gamma, \ \ e : A \cap B \ \ \vdash \ \ \Delta} \ (\cap_L) \qquad \frac{\Gamma \ \vdash \ \ r : A, \ \ \Delta \quad \Gamma \ \vdash \ \ r : B, \ \ \Delta}{\Gamma \ \vdash \ \ r : A \cap B, \ \ \Delta} \ (\cap_R)$$

$$\frac{\Gamma \ \vdash \ \Delta, \alpha : A}{\Gamma \ \vdash \ \Delta, \alpha : A \cup B} \ (\cup_{R-Var})$$

$$\frac{\Gamma, \ \ e : A \ \ \vdash \ \ \Delta \quad \Gamma, \ \ e : B \ \ \vdash \ \ \Delta}{\Gamma, \ \ e : A \cup B \ \ \vdash \ \ \Delta} \ (\cup_L) \qquad \frac{\Gamma \ \vdash \ \ r : A, \ \ \Delta}{\Gamma \ \vdash \ \ r : A \cup B, \ \ \Delta} \ (\cup_R)$$

The type system $\mathcal{M}^{\cap\cup}$

Rules $\cap_{L-Var}$ and $\cap_{R-Var}$ complicates reasoning about this system.

But applications of these rules can always be pushed toward the leaves of a typing tree.

In fact an equivalent formulation of the system removes these rules completely and replaces them with more flexible axiom schemes.

The system $\mathcal{M}^{\cap\cup}$ is obtained by replacing the rules $\cap_{L-Var}$ and $\cap_{R-Var}$ by more flexible rules e^{+}-ax and r^{+}-ax.

$$\frac{}{\Gamma,\ \alpha : A_i \ \vdash\ \alpha : A_1 \cup \cdots \cup A_n, \Delta}\ (e^{+}\text{-}ax) \qquad \frac{}{\Gamma, x : A_1 \cap \cdots \cap A_n \ \vdash\ x : A_i,\ \Delta}\ (r^{+}\text{-}ax)$$

$$\frac{\Gamma \ \vdash\ r : A,\ \Delta \quad \Gamma,\ e : B \ \vdash\ \Delta}{\Gamma,\ r \bullet e : A \longrightarrow B \ \vdash\ \Delta}\ (\longrightarrow L) \qquad \frac{\Gamma, x : A \ \vdash\ r : B,\ \Delta}{\Gamma \ \vdash\ \lambda x.r : A \longrightarrow B,\ \Delta}\ (\longrightarrow R)$$

$$\frac{c : (\Gamma, x : A \ \vdash\ \Delta)}{\Gamma,\ \tilde{\mu} x.c : A \ \vdash\ \Delta}\ (\tilde{\mu}) \qquad \frac{c : (\Gamma \ \vdash\ \alpha : A, \Delta)}{\Gamma \ \vdash\ \mu\alpha.c : A,\ \Delta}\ (\mu)$$

$$\frac{\Gamma \ \vdash\ r : A,\ \Delta \quad \Gamma,\ e : A \ \vdash\ \Delta}{\langle r \parallel e \rangle : (\Gamma \ \vdash\ \Delta)}\ (cut)$$

$$\frac{\Gamma,\ e : A \ \vdash\ \Delta}{\Gamma,\ e : A \cap B \ \vdash\ \Delta}\ (\cap_L) \qquad \frac{\Gamma \ \vdash\ r : A,\ \Delta \quad \Gamma \ \vdash\ r : B,\ \Delta}{\Gamma \ \vdash\ r : A \cap B,\ \Delta}\ (\cap_R)$$

$$\frac{\Gamma,\ e : A \ \vdash\ \Delta \quad \Gamma,\ e : B \ \vdash\ \Delta}{\Gamma,\ e : A \cup B \ \vdash\ \Delta}\ (\cup_L) \qquad \frac{\Gamma \ \vdash\ r : A,\ \Delta}{\Gamma \ \vdash\ r : A \cup B,\ \Delta}\ (\cup_R)$$

Subject reduction

Theorem Subject reduction:

If $c : \Gamma \vdash \Delta$ and if $c \longrightarrow c'$ then $c' : \Gamma \vdash \Delta$.

Strong normalization of typable terms: standard methods of proof fail!

The type of the term $\mu\alpha.\langle r \parallel e\rangle$ is the type of α.

It has nothing to do with the type of r and the type of e!

So standard methods based on reducibility break down.

Priorities

A priority π is a well-founded partial ordering defined on terms of the form $\mu\alpha.c$ and $\tilde{\mu}x.c$.

For example, the call-by-name priority is defined by

$$\tilde{\mu}x.c' \prec \mu\alpha.c$$

for all c and c', with no other relations.

The call-by-value priority is defined by

$$\mu\alpha.c \prec \tilde{\mu}x.c'$$

for all c and c', with no other relations.

Reductions and priorities

Let π be a priority.

- A reduction step respects π if
  - either its redex is not an instance of a critical pair,
  - or, letting the redex be $\langle \mu\alpha.c \parallel \widetilde{\mu}x.c' \rangle$, the $\widetilde{\mu}$-reduction is done if $\mu\alpha.c$ precedes $\widetilde{\mu}x.c'$, but the μ-reduction is done if $\mu x.c'$ precedes $\mu\alpha.c$.
- A reduction sequence is a π-reduction if each step respects π.
- A term t is strongly normalizing under π, or is a SN^{π}-term, if every π-reduction sequence out of t terminates.

$(S^\pi)^A$

Given a priority π, we define a set $(S^\pi)^A$ for each A.

This leads to two sets $(S^\pi)^A_r$ and $(S^\pi)^A_e$.

Lemma: If $r \in (S^\pi)^A_r$ and $e \in (S^\pi)^A_e$ then $\langle r \parallel e \rangle \in SN_\pi$.

Lemma: If t is typable of type A then $t \in (S^\pi)^A$.

Strong normalization of typable terms

Theorem: For every priority π, each term typable in $\mathcal{M}^{\cap\cup}$ is SN^{π}.

Corollary: Every typable term is strongly normalizing under both *call by name* and *call by value*.

Future work

- Finer analysis of reduction using intersection-union types: e.g., characterizing the normalizing strategies.
- “Real” programming: add arithmetic, conditional, fixpoints.
- Solving equations, i.e. unification.
- Denotational semantics.
- A calculus without active propositions the $\mathcal{X}$ calculus.