Vector Addition Tree Automata & Multiplicative Exponential Linear Logic

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September 24, 2004

Overview

- Introduction
- An example
- The automata side
- The linear logic side
- Bridging the two sides

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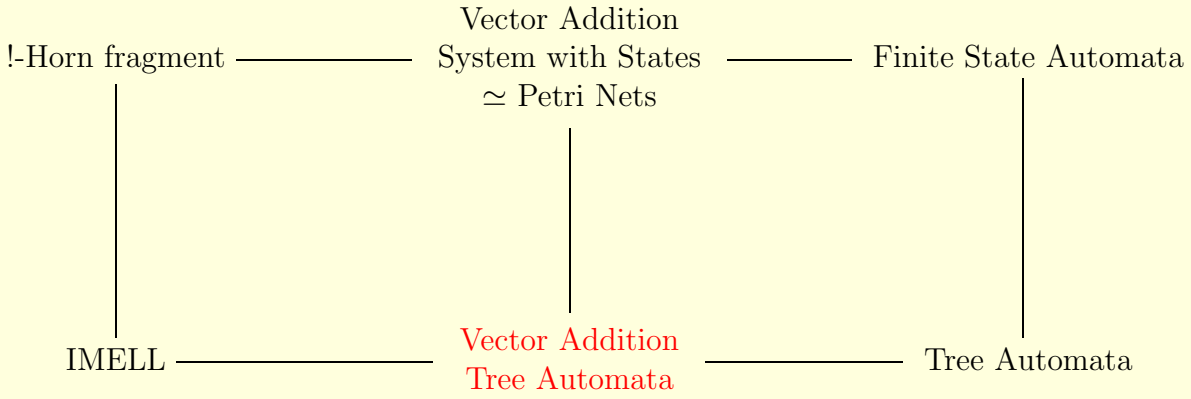

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Putting it as a (tree automata like) rewriting system:

$$\begin{array}{rcl} a & \to & q[(1,0)] \\ b & \to & q[(0,1)] \\ f(q[\mathbf{x}]) & \to & q[\mathbf{x} - (1,1)] \\ g(q[\mathbf{x}_1], q[\mathbf{x}_2]) & \to & q[\mathbf{x}_1 + \mathbf{x}_2] \end{array}$$

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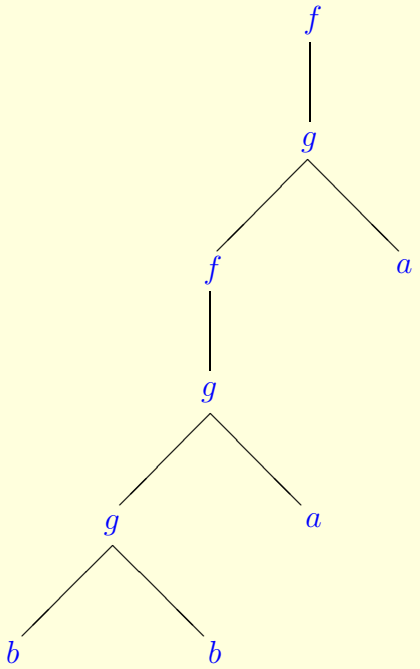

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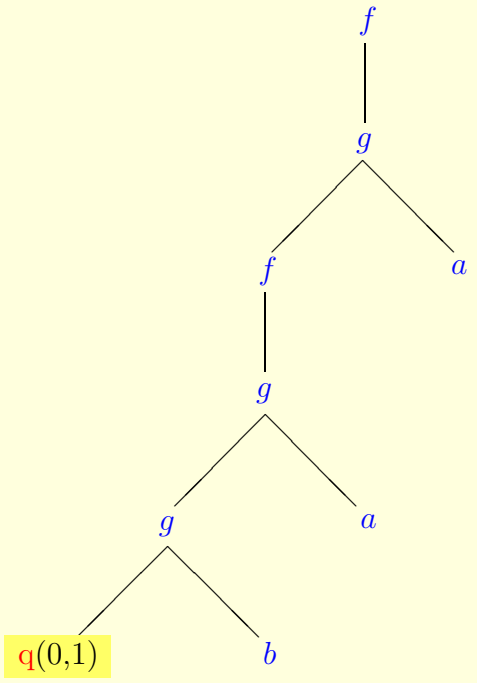

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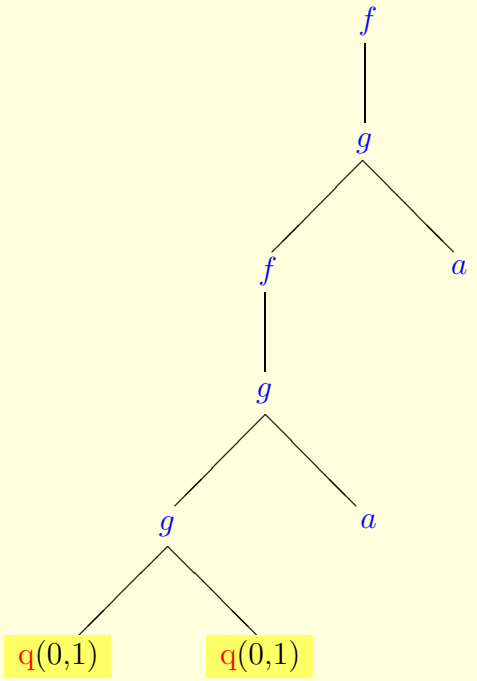

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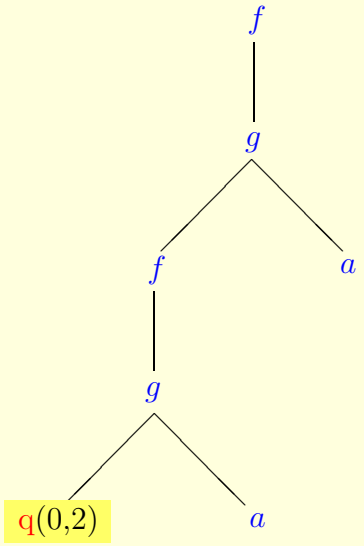

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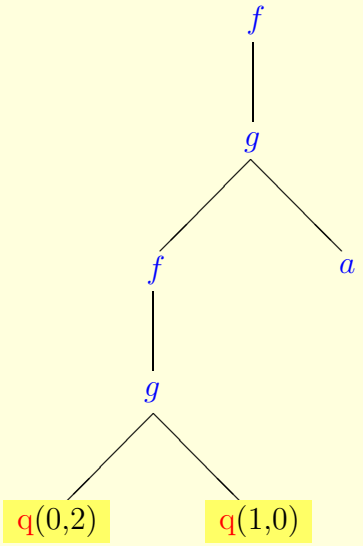

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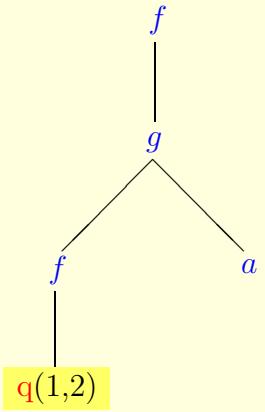

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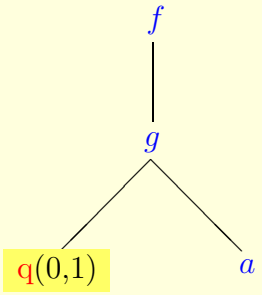

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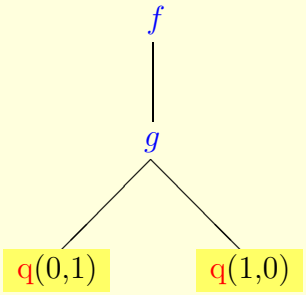

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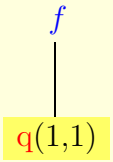

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Vector Addition Tree Automata

A k-VATA is a quadruple $\langle \mathcal{F}, Q, C_f, \Delta \rangle$ where:

- $\mathcal{F}$ is a ranked alphabet;
- Q is a finite set of states;
- C_f is a finite set of accepting configurations (i.e. elements of $Q \times \mathbb{N}^k$);
- Δ is a finite set of transition rules of the form:

$$f(q_0[\mathbf{x}_0], \ldots, q_{n-1}[\mathbf{x}_{n-1}]) \rightarrow q\left[\sum_{i \in n}(\mathbf{x}_i - \mathbf{c}_i) + \mathbf{c}\right]$$

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where:

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The rewriting relation is induced by the transition rules with the constraint that $\mathbf{x}_i - \mathbf{c}_i \in \mathbb{N}^k$ (this corresponds to the positivity condition in VASS).

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Proposition 1 *For any k-VATA $\mathcal{A}$ there exists a k-VATA $\mathcal{A}'$ in normal form such that $\mathcal{L}_{\mathcal{A}} = \emptyset$ iff $\mathcal{L}_{\mathcal{A}'} = \emptyset$.*

Linear Logic

IMELL

$$\mathcal{F} \;::=\; \mathbf{1} \mid \mathcal{A} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \multimap \mathcal{F} \mid\; !\mathcal{F}$$

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From IMELL to IMELL$_0$

Proposition 2 *IMELL is decidable iff IMELL$_0$ is decidable.*

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Proof

Let $\Gamma \vdash A$ an IMELL sequent.

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Add Σ, the following set of formulas to the antecedent of the sequent:

$!(p_F \multimap F^*)$,

$!(p_F \multimap p_F \otimes p_F)$,

$!(p_F \multimap \mathbf{1})$,

$!(p_{F_1} \multimap \cdots \multimap p_{F_n} \multimap p_F)$ whenever $p_{F_1} \multimap \cdots \multimap p_{F_n} \multimap F^*$ is provable.

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Let $\Gamma \vdash A$ an IMELL sequent.

Define $\Gamma^* \vdash A^*$ to be the sequent obtained by replacing each exponential subformula $!F$ by a fresh atomic proposition p_F.

Add Σ, the following set of formulas to the antecedent of the sequent:

$!(p_F \multimap F^*)$,
$!(p_F \multimap p_F \otimes p_F)$,
$!(p_F \multimap \mathbf{1})$,
$!(p_{F_1} \multimap \cdots \multimap p_{F_n} \multimap p_F)$ whenever $p_{F_1} \multimap \cdots \multimap p_{F_n} \multimap F^*$ is provable.

- If IMELL$_0$ is decidable then the construction is effective.

From IMELL to IMELL$_0$

Proposition 2 *IMELL is decidable iff IMELL$_0$ is decidable.*

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- If IMELL$_0$ is decidable then the construction is effective.
- $\Gamma \vdash A$ is provable if and only if $\Sigma, \Gamma^* \vdash A^*$ is provable. □

From $\mathbf{IMELL_0}$ to $\mathbf{IMELL_0^{-\circ}}$

Proposition 3 *$IMELL_0$ is decidable iff $IMELL_0^{-\circ}$ is decidable.*

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Proof Let b a fresh atomic proposition, we define two translation on multiplicative formulas:

$A^+ = A^- \multimap b$, for any formula A

$\mathbf{1}^- = b$

$a^- = a \multimap b$

$(A \otimes B)^- = A^+ \multimap (B^+ \multimap b)$

$(A \multimap B)^- = (A^+ \multimap B^+) \multimap b$

and extend it to IMELL_0 with $(!A)^+ = !(A^+)$.

From IMELL_0 to $\mathrm{IMELL}_0^{-\circ}$

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- $\Gamma \vdash A$ an IMELL_0 sequent is provable iff $\Gamma^+ \vdash A^+$ is provable. □

From $\mathbf{IMELL}_0^{-\circ}$ to $\mathbf{s\text{-}IMELL}_0^{-\circ}$

Proposition 4 *$IMELL_0^{-\circ}$ is decidable iff s-$IMELL_0^{-\circ}$ is decidable.*

From $\mathbf{IMELL}_0^{-\circ}$ to s-$\mathbf{IMELL}_0^{-\circ}$

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Proof

Lemma 1 *Let $\Gamma \vdash A$ an IMELL sequent.*

$$\underbrace{\ldots F \ldots F \ldots}_{\Gamma} \vdash \underbrace{\ldots F \ldots}_{A} \iff !(p \multimap F), !(F \multimap p), \underbrace{\ldots p \ldots p \ldots}_{\Gamma} \vdash \underbrace{\ldots p \ldots}_{A}$$

From $\text{IMELL}_0^{-\circ}$ to $\text{s-IMELL}_0^{-\circ}$

Proposition 4 *$\text{IMELL}_0^{-\circ}$ is decidable iff $\text{s-IMELL}_0^{-\circ}$ is decidable.*

Proof

Lemma 1 *Let $\Gamma \vdash A$ an IMELL sequent.*

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$$!\Sigma, \Gamma \vdash A \text{ is a provable } \text{IMELL}_0^{-\circ} \text{ sequent}$$
$$\Updownarrow$$
$$!\Sigma', \underbrace{\Gamma'}_{atomic} \vdash a \text{ is a provable } \text{IMELL}_0^{-\circ} \text{ sequent}$$
$$\Updownarrow$$
$$! \underbrace{\Sigma''}_{\leq 2\text{`}\multimap\text{`}}, \underbrace{\Gamma'}_{atomic} \vdash a \text{ is a provable } \text{IMELL}_0^{-\circ} \text{ sequent}$$

□

From VATA to IMELL

Let $\mathcal{A} = \langle \mathcal{F}, Q, \{(q_f, \mathbf{0})\}, \Delta \rangle$ a k-VATA in normal form.

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We define the set of atomic types $A = Q \cup \{a_0, \ldots, a_{k-1}\}$ and Σ by:

$$\begin{array}{rcl} \Delta & & \Sigma \\ f \longrightarrow q[\mathbf{e}_i] & \rightsquigarrow & a_i \multimap q \\ f(q_0[\mathbf{x}_0]) \longrightarrow q[\mathbf{x}_0 - \mathbf{e}_i] & \rightsquigarrow & (a_i \multimap q_0) \multimap q \\ f(q_0[\mathbf{x}_0], q_1[\mathbf{x}_1]) \longrightarrow q[\mathbf{x}_0 + \mathbf{x}_1] & \rightsquigarrow & q_0 \multimap q_1 \multimap q \end{array}$$

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Proposition 5 $\mathcal{L}(\mathcal{A}) \neq \varnothing$ *iff* $!\Sigma \vdash q_f$.

From IMELL to VATA

Let $!\Sigma, \Gamma \vdash a_0$ an s-IMELL$_0^{\neg\circ}$ sequent and $\{a_0, \ldots, a_{k-1}\}$ an enumeration of the atomic formulas of the sequent.

From IMELL to VATA

Let $!\Sigma, \Gamma \vdash a_0$ an s-IMELL$_0^{\multimap}$ sequent and $\{a_0, \ldots, a_{k-1}\}$ an enumeration of the atomic formulas of the sequent. We define the set of state $Q = \{q_0, \ldots, q_{k-1}\}$, the only final configuration $(q_0, |\Gamma|)$ and the set of transitions:

$$
\begin{array}{rcl}
\Sigma & & \Delta \\
 & \rightsquigarrow & c_i \to q_i[\mathrm{e}_i] \\
a_j \multimap a_l & \rightsquigarrow & f(q_j[\mathbf{x}]) \to q_l[\mathbf{x}] \\
(a_j \multimap a_l) \multimap a_m & \rightsquigarrow & g(q_l[\mathbf{x}]) \to q_m[\mathbf{x} - \mathrm{e}_j] \\
a_j \multimap a_l \multimap a_m & \rightsquigarrow & h(q_j[\mathbf{x}_0], q_l[\mathbf{x}_1]) \to q_m[\mathbf{x}_0 + \mathbf{x}_1]
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\end{array}
$$

Proposition 6 *$!\Sigma, \Gamma \vdash a_0$ is provable in s-IMELL$_0^{-\circ}$ iff $\mathcal{L}(\mathcal{A}) \neq \varnothing$.*

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- Future work:

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