

Towards a denotational semantics for the ρ -calculus

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Menu of the next 30 minutes

- ④ Presentation of the ρ -calculus (a fragment).
- ④ Scott semantics.
- ④ Discussions (weakness & new insights).
- ④ On the horizon.

The ρ -calculus as a generalization of λ -calculus

- ⊙ “Main design concept: to make all the basic ingredients of rewriting explicit objects” (from IGPAL-01)

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Presentation of the ρ -calculus

REMINDERS+EXAMPLES

EXAMPLES

SKIP

Syntax of the ρ -calculus

Terms	M, N	$::=$	x	(Variables)	
				c	(Constructors)
				$\lambda P . M$	(Abstraction)
				$M \bullet N$	(Functional application)
				$[P \ll N]M$	(Delayed matching constraints)
				$M; N$	(Structure)
Patterns	P	$::=$	x	(Variables)	
				$cP_1 \dots P_n$	(Constructors)

Semantics of the ρ -calculus

$$(\rho) \quad (\lambda P . M) \bullet N \quad \rightarrow \quad [P \ll N] \bullet M$$

$$(\sigma) \quad [P \ll N] \bullet M \quad \rightarrow \quad \sigma M$$

if $\sigma = P \Leftarrow N$

$$(\delta) \quad (M_1 ; M_2) \bullet N \quad \rightarrow \quad M_1 \bullet N ; M_2 \bullet N$$

Examples

$(\lambda \text{ cons}(x, l) . x) \bullet \text{cons}(1, \text{nil})$

$\mapsto_{\rho} [\text{cons}(x, l) \ll \text{cons}(1, \text{nil})]x$

$\mapsto_{\sigma} 1$

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$$(\lambda \odot . \text{stop}; \lambda \odot . \text{go}; \lambda \odot . \text{go}; \lambda \odot . \text{stop}) \bullet \odot$$

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Theory on structures?

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Computation equivalence not always clear without the Church-Rosser property
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 - Syntactic corollaries (Typically: *terms foo and bar are not convertible*)

Scott domains

- A **Scott domain** is a poset (D, \leq) satisfying particular axioms.
Namely: CPO + bottom element + bounded completeness + algebraicity
 - Work with **continuous functions** (induced by **Scott topology** on D)
 - Domain equations such as $D \approx (D \rightarrow D)$ have non-trivial solutions
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 - Continuity: \equiv Monotonicity + Finiteness [i.e. commutation with directed limits]

Rho-models

A ρ -model is a Scott-domain D equipped with:

© **Beta-rule** Two (Scott-continuous) functions:

$$\text{lam} : (D \rightarrow D) \rightarrow D$$

$$\text{app} : D \rightarrow (D \rightarrow D)$$

s.t.

$$\text{app} \circ \text{lam} = \text{id}_{D \rightarrow D}$$

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⊙ **Pattern-matching** For each constructor c of arity n , two functions:

$$\begin{array}{ll} c_* & : D^n \rightarrow D \\ c^* & : D \rightarrow \text{opt}(D^n) \end{array} \quad \text{s.t.} \quad c^* \circ c_* = (\vec{w} \mapsto \text{some}(\vec{w}))$$

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$$\text{error}_P : D \times (D^n \rightarrow D) \rightarrow D \quad (\text{no axioms})$$

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⊙ **Structures** A function:

$$\text{merge} : D \times D \rightarrow D \quad \text{s.t.} \quad \begin{array}{l} \text{app}(\text{merge}(v_1, v_2), w) = \\ \text{merge}(\text{app}(v_1, w), \text{app}(v_2, w)) \end{array}$$

Additional axioms

Extensions of the basic ρ -theory require extra axioms:

⊙ **Extensionality:**

(η -rule)

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⊙ **ACI theory** (or any combination of the axioms **A**, **C**, **I**) :

$$(\text{Associativity}) \quad \text{merge}(\text{merge}(v_1, v_2), v_3) = \text{merge}(v_1, \text{merge}(v_2, v_3))$$

$$(\text{Commutativity}) \quad \text{merge}(v_1, v_2) = \text{merge}(v_2, v_1)$$

$$(\text{Idempotence}) \quad \text{merge}(v, v) = v$$

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(Idempotence) $\text{merge}(v, v) = v$

⊙ **Constructor discrimination :**

for all $c \neq c'$ $c^* \circ c'_* = (- \mapsto \text{none})$

Compiling pattern matching

© Compositionality of pattern-matching

The interpretation of constructors (of arity n) provided by any ρ -model

$$c_* : D^n \rightarrow D, \quad c^* : D \rightarrow \text{opt}(D^n) \quad \text{s.t.} \quad c^* \circ c_* = (\vec{w} \mapsto \text{some}(\vec{w}))$$

is easily extended to all ML-style patterns P (of arity n):

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© The matching function $\text{match}_P : D \times (D^n \rightarrow D) \rightarrow D$ is defined by

$$\text{match}_P(v, f) = \text{case_opt } P^*(v) \text{ with}$$

	$\text{some}(\vec{w})$	\mapsto	$f(\vec{w})$
	none	\mapsto	$\text{error}_P(v, f)$

(where case_opt is the destruction operation of values of type $\text{opt}(D^n)$)

The interpretation function

© **Valuations** A **valuation** is a function $\rho : \mathcal{V} \rightarrow D$ (\mathcal{V} = set of all variables)
The set of all valuations $D^{\mathcal{V}}$ is a Scott-domain (i.e. infinite cartesian product).

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$$\llbracket x \rrbracket_{\rho} = \rho(x)$$

$$\llbracket c \rrbracket_{\rho} = c_*$$

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$$\llbracket [P_{\vec{x}} \ll N] M \rrbracket_{\rho} = \text{match}_P(\llbracket N \rrbracket_{\rho}, (\vec{w} \mapsto \llbracket M \rrbracket_{(\rho; \vec{x} \leftarrow \vec{w})}))$$

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⊙ **Closed case** If M is closed, we write $\llbracket M \rrbracket = \llbracket M \rrbracket_{\rho}$ (does not depend on ρ)

The soundness property

Write: $D \models M_1 = M_2 \quad \equiv \quad \forall \rho \in D^{\mathcal{V}} \quad \llbracket M_1 \rrbracket_{\rho}^D = \llbracket M_2 \rrbracket_{\rho}^D$

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- Soundness still holds for any combination of A, C, I and/or the η -reduction rule (provided we restrict to the corresponding notion of ρ -model).

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Remarks:

- Soundness still holds for any combination of A, C, I and/or the η -reduction rule (provided we restrict to the corresponding notion of ρ -model).
- Proofs **do not depend** on any kind of **Church-Rosser/confluence** property.

A fundamental particular case: D^∞

Let D^∞ be the 'historical' non-trivial solution of: $D^\infty \approx (D^\infty \rightarrow D^\infty)$

and set: $\text{merge}(v_1, v_2) := \sup\{v_1, v_2\}$ [Works since D^∞ has a top-element]
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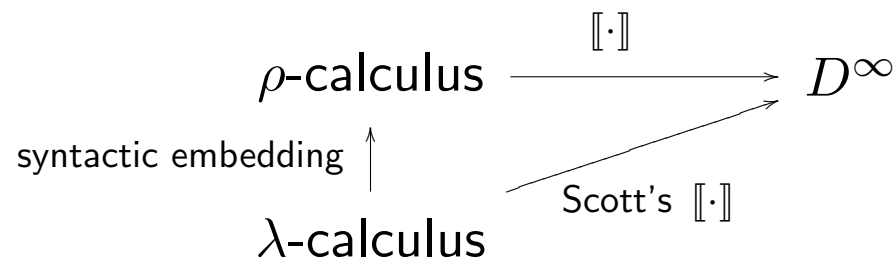
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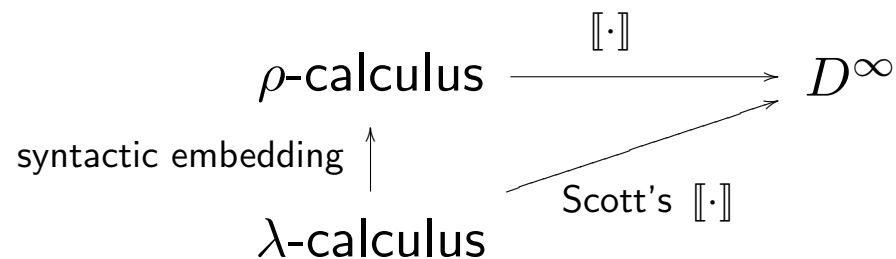


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Since Scott's interpretation is faithful (i.e. injective) on $\beta\eta$ -normal forms, we get:

Corollary (Conservativity on λ -normal forms / Weak C.R.) — *The $\rho\sigma\delta\eta$ ACI-theory of the ρ -calculus identifies no pair of distinct $\beta\eta$ -normal terms of the λ -calculus.*

Discussion

- ⊙ Compositionality of matching.
- ⊙ Right distributivity rule.
- ⊙ Notion of values.
- ⊙ Weakness of the model
 - ⊙ Management of errors.
 - ⊙ Structures (cannot be destructed).
⇒ **Monads**.

Another perspective

- The construction $\lambda P_{\vec{x}} . M_{\vec{x}}$ can be understood as $M \circ P^{-1}$

The (hidden) dream of ρ -calculists: allow the use of all terms as patterns...

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- But **inversion** is fundamentally **anti-monotonic**! [Think of $x \mapsto 1/x$, $f \mapsto f^{-1}$]

\Rightarrow The full dream of ρ -calculists will not be realised with Scott-style semantics

\Rightarrow But an exciting challenge for denotational semantics!