Intersection Types à la Church

INTER

$$\lambda x.M \;\; vs. \;\; \lambda x{:}\boxed{\,?\,}.M$$

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INTER

Motivations

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- Design a satisfying typed version of the intersection type assignment system discipline *à la* Curry of Coppo Dezani ($\Lambda^{u}\wedge$), late '70's

- Since the TCS note of Hindley "Coppo Dezani Types do not correspond to propositional logic", the problem is "open"

- Many researcher try our chance .... Reynolds, Pierce, Venneri *et al.*, Ronchi, Roversi, Wells *et al.*, and many more ...

- Curry-Howard isomorphism and proof-theory is a natural use, type systems for PL is another wish

- There is no general consensus on what is "the good" Church version

- "Fresh" competition for a longstanding problem

INTER

Contents

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- We design a fully decorated lambda calculus *à la* Church ($\Lambda\wedge$) based on the well-known intersection type assignment system discipline *à la* Curry of Coppo Dezani ($\Lambda^{\mathsf{u}}\wedge$), late '70's

- As any system *à la* Church it enjoy the Curry-Howard isomorphism, property that lack in the original type assignment, as remarked by Hindley in the middle of '80s

- This system could be considered as an alternative target calculus for typed intermediate languages in compilers or as an alternative to classical polymorphic lambda calculi *à la* Girard's system F

- The system could be extended with a classical subtyping relation in the style of other presentations. A simple type reconstruction algorithm is presented. Last but not least, $\Lambda\wedge$'s metatheory is not only feasible but easy

The Reference *à la* Curry

$$M \;::=\; x \mid \lambda x.M \mid M\,M$$

$$\sigma \;::=\; \alpha \mid \sigma\!\rightarrow\!\sigma \mid \sigma \wedge \sigma$$

$$\frac{x{:}\sigma \in \Gamma}{\Gamma \vdash_\wedge x : \sigma}\;(Var) \qquad \frac{\Gamma, x{:}\sigma_1 \vdash_\wedge M : \sigma_2}{\Gamma \vdash_\wedge \lambda x.M : \sigma_1\!\rightarrow\!\sigma_2}\;(\rightarrow I)$$

$$\frac{\begin{array}{l}\Gamma \vdash_\wedge M : \sigma_1\!\rightarrow\!\sigma_2\\ \Gamma \vdash_\wedge N : \sigma_1\end{array}}{\Gamma \vdash_\wedge M\,N : \sigma_2}\;(\rightarrow E) \qquad \frac{\begin{array}{c}\Gamma \vdash_\wedge M : \sigma_1\\ \Gamma \vdash_\wedge M : \sigma_2\end{array}}{\Gamma \vdash_\wedge M : \sigma_1 \wedge \sigma_2}\;(\wedge I)$$

$$\frac{\Gamma \vdash_\wedge M : \sigma_1 \wedge \sigma_2}{\Gamma \vdash_\wedge M : \sigma_1}\;(\wedge E_L) \qquad \frac{\Gamma \vdash_\wedge M : \sigma_1 \wedge \sigma_2}{\Gamma \vdash_\wedge M : \sigma_2}\;(\wedge E_R)$$

The Problems

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \vdots \\ \Gamma \vdash_\wedge M : \sigma_1\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ \vdots \\ \Gamma \vdash_\wedge M : \sigma_2\end{array}}{\Gamma \vdash_\wedge M : \sigma_1 \wedge \sigma_2} \; (\wedge I)$$

- *Hic 1*: A typed M must record two proofs $\mathcal{D}_1$, and $\mathcal{D}_2$, while by Curry-Howard every term record one proof

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- *Hic 2*: A Church system must be syntax directed, while $(\wedge I), (\wedge E_L), (\wedge E_R)$ are not. This disconnect the λ-term with its derivation (hence loose Curry-Howard)

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- *Hic 4*: Anything else? ...

The Problems

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- *Hic 1*: A typed M must record two proofs $\mathcal{D}_1$, and $\mathcal{D}_2$, while by Curry-Howard every term record one proof
- *Hic 2*: A Church system must be syntax directed, while $(\wedge I), (\wedge E_L), (\wedge E_R)$ are not. This disconnect the λ-term with its derivation (hence loose Curry-Howard)
- *Hic 3*: By type-erasure, we must refind the original "star system *à la* Curry"
- *Hic 4*: Anything else? ... yes, it must be sexy and easy to understand

An Example of the Problem

$$\dfrac{\dfrac{x{:}\sigma_1 \vdash_\wedge x : \sigma_1}{\vdash_\wedge \lambda x.x : \sigma_1 \to \sigma_1}\,(\to I) \qquad \dfrac{x{:}\sigma_2 \vdash_\wedge x : \sigma_2}{\vdash_\wedge \lambda x.x : \sigma_2 \to \sigma_2}\,(\to I)}{\vdash_\wedge \lambda x.x : (\sigma_1 \to \sigma_1) \wedge (\sigma_2 \to \sigma_2)}\,(\wedge I)$$

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untypable using a naïve corresponding rule *à la* Church for $(\wedge I)$

$$\dfrac{\dfrac{x{:}\sigma_1 \vdash_\wedge x : \sigma_1}{\vdash_\wedge \lambda x{:}\sigma_1.x : \sigma_1 \to \sigma_1}\,(\to I) \qquad \dfrac{x{:}\sigma_2 \vdash_\wedge x : \sigma_2}{\vdash_\wedge \lambda x{:}\sigma_2.x : \sigma_2 \to \sigma_2}\,(\to I)}{\vdash_\wedge \lambda x{:}\boxed{?}.x : (\sigma_1 \to \sigma_1) \wedge (\sigma_2 \to \sigma_2)}\,(\wedge I)$$

General Requirements Relating Curry's and Church's Systems

1. there exists an erasing function $\mathcal{J}$ erasing type information from terms, such that, if M is a typed term, then $\mathcal{J}(M) \in \Lambda$

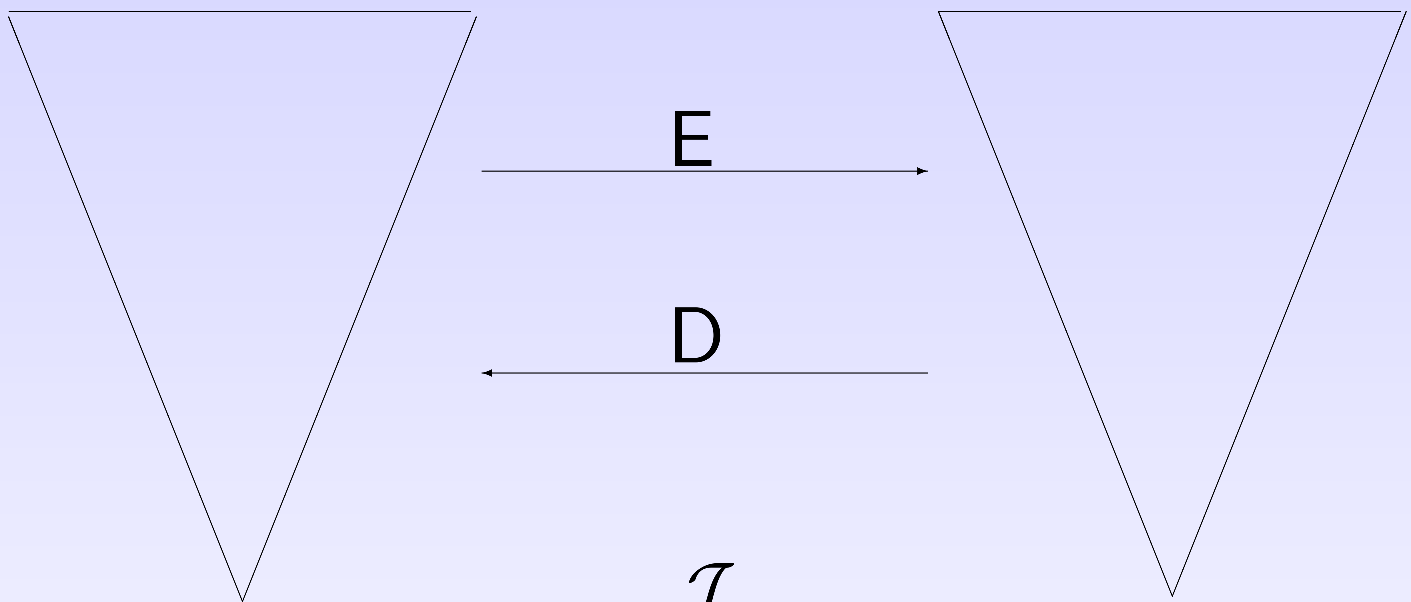
2. typed and type assignment proofs are *isomorphic*, $i.e.$ both the application of the *erasing function* $\mathcal{J}$ on all terms in a typed proofs produces a correct type assignment proof, and every type assignment proof is obtained from a typed one by erasure. This is done via:

   (a) an erasing function E erasing type informations from typed derivations such that if $\mathcal{D}$ is a derivation in $\Lambda\wedge$, then $\mathsf{E}(\mathcal{D})$ is a derivation in $\Lambda^{\mathsf{u}}\wedge$, and

   (b) a decorating function D on untyped derivations such that if $\mathcal{D}'$ is a derivation in $\Lambda^{\mathsf{u}}\wedge$, then $\mathsf{D}(\mathcal{D}')$ is a derivation in $\Lambda\wedge$,

   (c) a suitable relation between E and D, $i.e.$: $\mathsf{D} \circ \mathsf{E} = \mathsf{I} = \mathsf{E} \circ \mathsf{D}$, where I is the identity function

Graphically Speaking



$$\Gamma \vdash M@\Delta : \sigma \quad \underset{\mathcal{J}^{-1}}{\overset{\mathcal{J}}{\rightleftarrows}} \quad \mathcal{J}(\Gamma) \vdash_{\wedge} \mathcal{J}(M@\Delta) : \sigma$$

General Requirements Relating Curry's and Church's Systems

Moreover, we want that the intersection calculus *à la* Church inherits all the requirements of intersection calculus *à la* Curry, namely:

3. subject reduction

4. strong normalization of typable terms

plus the following ones:

5. unicity of typing

6. decidability of type reconstruction and of type checking

No one of the proposals present in the literature satisfies all the given requirements. The typed languages proposed in REY and PIE are not complete with respect to the type assignment, the ones in CAPV, RNK and CRYPT1 do not satisfy requirement 1, while the language in CRYPT2 does not satisfy requirement 2

Our Recipe/Our syntax

- Introduction of a (imperative) notion of tree-store Δ that remember associations between locations and types and record the shape of a proof, *i.e.* the $\Lambda\mathcal{D}$ calculus

$$\begin{array}{rcll} \Delta & ::= & \iota \mid \lambda\iota{:}\sigma.\Delta \mid \Delta\,\Delta & \text{(for syntax directed rules)} \\ & & \Delta \wedge \Delta \mid \swarrow\!\Delta \mid \Delta\!\searrow & \text{(for intersection rules)} \end{array}$$

- Decoration of bound variables with type-locations, *i.e.* the $\Lambda\wedge$ calculus

$$M ::= x \mid \lambda x@\iota.M \mid M\,M$$

- "Linkage" of $\Lambda\wedge$ and $\Lambda\mathcal{D}$ via a suitable notion of context, and judgment, *i.e.*

$$\Gamma ::= \epsilon \mid \Gamma, x@\iota{:}\sigma \qquad \Gamma \vdash M@\Delta : \sigma$$

Before I tell you the full story ...

Curry

$$\dfrac{\dfrac{x{:}\sigma_1 \vdash_\wedge x : \sigma_1}{\vdash_\wedge \lambda x.x : \sigma_1{\rightarrow}\sigma_1}\,(\rightarrow I) \qquad \dfrac{x{:}\sigma_2 \vdash_\wedge x : \sigma_2}{\vdash_\wedge \lambda x.x : \sigma_2{\rightarrow}\sigma_2}\,(\rightarrow I)}{\vdash_\wedge \lambda x.x : (\sigma_1{\rightarrow}\sigma_1) \wedge (\sigma_2{\rightarrow}\sigma_2)}\,(\wedge I)$$

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Church

$$\dfrac{\dfrac{\dfrac{x \quad :\sigma_1 \in x \quad :\sigma_1}{x \quad :\sigma_1 \vdash x \quad : \sigma_1}(Var)}{\vdash (\lambda x \quad .x) \quad : \sigma_1 \to \sigma_1}(\to I) \quad \dfrac{\dfrac{x \quad :\sigma_2 \in x \quad :\sigma_2}{x \quad :\sigma_2 \vdash x \quad : \sigma_2}(Var)}{\vdash (\lambda x \quad .x) \quad : \sigma_2 \to \sigma_2}(\to I)}{\vdash (\lambda x \quad .x) \quad : (\sigma_1 \to \sigma_1) \wedge (\sigma_2 \to \sigma_2)}(\wedge I)$$

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INTER

One Step Semantics

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$$(\lambda x@\iota.M)\,N \quad \rightarrow_\beta \quad M[N/x]$$ for λ-terms

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$(\lambda x@\iota.M)\, N$	$\longrightarrow_{\beta}$	$M[N/x]$	for λ-terms
$(\lambda\iota{:}\sigma.\Delta_1)\, \Delta_2$	$\longrightarrow_{\iota}$	$\Delta_1[\Delta_2/\iota]$	for tree-stores
$\swarrow(\Delta_1 \wedge \Delta_2)$	$\longrightarrow_{\pi_1}$	Δ_1	...
$(\Delta_1 \wedge \Delta_2)\searrow$	$\longrightarrow_{\pi_2}$	Δ_2	...

One Step Semantics

$(\lambda x@\iota.M)\,N$	$\rightarrow_\beta$	$M[N/x]$	for λ-terms
$(\lambda\iota{:}\sigma.\Delta_1)\,\Delta_2$	$\rightarrow_\iota$	$\Delta_1[\Delta_2/\iota]$	for tree-stores
$\swarrow(\Delta_1 \wedge \Delta_2)$	$\rightarrow_{\pi_1}$	Δ_1	$\ldots$
$(\Delta_1 \wedge \Delta_2)\searrow$	$\rightarrow_{\pi_2}$	Δ_2	$\ldots$
$(\lambda x@\iota.M)@\Delta$	$\rightarrow_\alpha$	$(\lambda y{:}\iota.M[y/x])@\Delta$	if $y \notin \mathsf{Fv}(M)$
$M@(\lambda\iota_1{:}\sigma.\Delta)$	$\rightarrow_{\alpha_\iota}$	$M[\iota_2/\iota_1]@(\lambda\iota_2{:}\sigma.\Delta[\iota_2/\iota_1])$	if $\iota_2 \notin \mathsf{Fv}(\Delta)$

Define $\mapsto_\beta, \mapsto\!\!\!\!\rightarrow_\beta, =_\beta, \rightarrow_{\iota\pi}, \mapsto_{\iota\pi}, \mapsto\!\!\!\!\rightarrow_{\iota\pi}, =_{\iota\pi}$, $\mathsf{Fv}(\Delta)$ as usual ...

INTER

Types

Type Rules (I)

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma}\ (Var)$$

$$\frac{\Gamma, x : \sigma_1 \vdash M : \sigma_2}{\Gamma \vdash (\lambda x . M) : \sigma_1 \to \sigma_2}\ (\to I)$$

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Type Rules (II)

$$\frac{\Gamma \vdash M : \sigma_1 \quad \Gamma \vdash M : \sigma_2}{\Gamma \vdash M : \sigma_1 \wedge \sigma_2} \; (\wedge I)$$

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$$\frac{\Gamma \vdash M@\Delta_1 : \sigma_1 \quad \Gamma \vdash M@\Delta_2 : \sigma_2}{\Gamma \vdash M@(\Delta_1 \wedge \Delta_2) : \sigma_1 \wedge \sigma_2} \ (\wedge I)$$

$$\frac{\Gamma \vdash M@\Delta : \sigma_1 \wedge \sigma_2}{\Gamma \vdash M@(\swarrow\Delta) : \sigma_1} \ (\wedge E_L)$$

$$\frac{\Gamma \vdash M@\Delta : \sigma_1 \wedge \sigma_2}{\Gamma \vdash M@(\Delta\searrow) : \sigma_2} \ (\wedge E_R)$$

INTER:

Examples

The 1 Term

Let $\Gamma \triangleq x@0{:}\sigma_1{\to}\sigma_2, y@1{:}\sigma_1$. We show a monomorphic derivation for

$$(\lambda x@0.\lambda y@1.x\,y)@(\lambda 0{:}\sigma_1{\to}\sigma_2.\lambda 1{:}\sigma_1.0\,1)$$

$$\dfrac{\dfrac{\Gamma \vdash x@0 : \sigma_1{\to}\sigma_2 \quad \Gamma \vdash y@1 : \sigma_1}{\Gamma \vdash (x\,y)@(0\,1) : \sigma_2}\,(\to E)}{\dfrac{x@0{:}\sigma_1{\to}\sigma_2 \vdash (\lambda y@1.x\,y)@(\lambda 1{:}\sigma_1.0\,1) : \sigma_1{\to}\sigma_2}{\vdash (\lambda x@0.\lambda y@1.x\,y)@(\lambda 0{:}\sigma_1{\to}\sigma_2.\lambda 1{:}\sigma_1.0\,1) : (\sigma_1{\to}\sigma_2){\to}\sigma_1{\to}\sigma_2}}\,(\to I)$$

Since only syntax directed rules are used, the lambda-term $(\lambda x@0.\lambda y@1.x\,y)$ faithfully encode its proof

The Polymorphic Self-application

Let $\Gamma \triangleq x@0{:}\sigma_2$, with $\sigma_2 \equiv (\sigma_1{\to}\sigma_1) \wedge \sigma_1$. We show a polymorphic derivation for the classical self-application

$$\lambda x@0.x\,x \quad \text{in the tree-store} \quad \Delta \triangleq \lambda 0{:}\sigma_2.(\swarrow 0)\,(0\searrow)$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash x@0 : \sigma_2}{\Gamma \vdash x@(\swarrow 0) : \sigma_1{\to}\sigma_1}\,(\wedge E_L) \qquad \dfrac{\Gamma \vdash x@0 : \sigma_2}{\Gamma \vdash x@(0\searrow) : \sigma_1}\,(\wedge E_R)}{\Gamma \vdash (x\,x)@(\swarrow 0)\,(0\searrow) : \sigma_1}\,(\to E)}{\vdash (\lambda x@0.x\,x)@(\lambda 0{:}\sigma_2.(\swarrow 0)\,(0\searrow)) : \sigma_2 \to \sigma_1}\,(\to I)$$

Note how the tree-store Δ memorize exactly the skeleton of the derivation while the self-application loose the proof structure

A Self-application's Application

We show a derivation assigning the following types to term

$$(\lambda x@0.x\,x)(\lambda x@1.x) \qquad \sigma_1 \triangleq \sigma{\rightarrow}\sigma \qquad \sigma_2 \equiv (\sigma_1{\rightarrow}\sigma_1) \wedge \sigma_1$$

$$\Delta \triangleq \lambda 0{:}\sigma_2.(\swarrow 0)\,(0 \searrow) \qquad \Delta_1 \triangleq \lambda 1{:}\sigma_1.1 \qquad \Delta_2 \triangleq \lambda 1{:}\sigma_2.1$$

$$\dfrac{\dfrac{\text{as in poly-self-appl}}{\vdash (\lambda x@0.x\,x)@\Delta : \sigma_2{\rightarrow}\sigma_1} \qquad \dfrac{\begin{array}{l}\text{as in poly-id}\\ \vdash (\lambda x@1.x)@\Delta_1 : \sigma_1{\rightarrow}\sigma_1\\ \vdash (\lambda x@1.x)@\Delta_2 : \sigma_1\end{array}}{\vdash (\lambda x@1.x)@(\Delta_1 \wedge \Delta_2) : \sigma_2}}{\vdash ((\lambda x@0.x\,x)\,(\lambda x@1.x))@\Delta\,(\Delta_1 \wedge \Delta_2) : \sigma_1}$$

INTER:

Isomorphism

The Function $\mathcal{J}$

$$
\begin{array}{rcl}
\mathcal{J}(x@\_) & \triangleq & x \\
\mathcal{J}((\lambda x@\iota.M)@\_) & \triangleq & \lambda x.\mathcal{J}(M@\_) \\
\mathcal{J}((M\,N)@\_) & \triangleq & \mathcal{J}(M@\_)\,\mathcal{J}(N@\_) \\
\mathcal{J}(\epsilon) & \triangleq & \epsilon \\
\mathcal{J}(\Gamma, x@\iota{:}\sigma) & \triangleq & \mathcal{J}(\Gamma), x{:}\sigma
\end{array}
$$

The Function E (I)

$$\mathsf{E}\left(\frac{x@\iota{:}\sigma \in \Gamma}{\Gamma \vdash x@\iota : \sigma}\ (Var) \right) \triangleq \begin{cases} \dfrac{x{:}\sigma \in \Gamma}{\Gamma' \vdash_\wedge x : \sigma}\ (Var) \\ \mathcal{J}(\Gamma) \equiv \Gamma' \end{cases}$$

$$\mathsf{E}\left(\frac{\Gamma, x@\iota{:}\sigma_1 \vdash M@\Delta : \sigma_2}{\Gamma \vdash (\lambda x@\iota.M)@(\lambda\iota{:}\sigma_1.\Delta) : \sigma_1 \to \sigma_2}\ (\to I) \right) \triangleq \begin{cases} \dfrac{\mathsf{E}(\mathcal{D}) : \Gamma', x{:}\sigma_1 \vdash_\wedge M' : \sigma_2}{\Gamma' \vdash_\wedge \lambda x.M' : \sigma_1 \to \sigma_2}\ (\to I) \\ \mathcal{J}(\Gamma, x@\iota{:}\sigma_1) \equiv \Gamma', x{:}\sigma_1\ \& \\ \mathcal{J}(M@\Delta) \equiv M' \end{cases}$$

$$\mathsf{E}\left(\frac{\begin{array}{l}\mathcal{D}_1 : \Gamma \vdash M@\Delta_1 : \sigma_1 \to \sigma_2 \\ \mathcal{D}_2 : \Gamma \vdash N@\Delta_2 : \sigma_1\end{array}}{\Gamma \vdash (M\ N)@(\Delta_1\ \Delta_2) : \sigma_2}\ (\to E) \right) \triangleq \begin{cases} \dfrac{\begin{array}{l}\mathsf{E}(\mathcal{D}_1) : \Gamma' \vdash_\wedge M' : \sigma_1 \to \sigma_2 \\ \mathsf{E}(\mathcal{D}_2) : \Gamma' \vdash_\wedge N' : \sigma_1\end{array}}{\Gamma' \vdash_\wedge M'\ N' : \sigma_2}\ (\to E) \\ \mathcal{J}(\Gamma) \equiv \Gamma'\ \& \\ \mathcal{J}(M@\Delta_1) \equiv M'\ \&\ \mathcal{J}(N@\Delta_2) \equiv N' \end{cases}$$

The Function E (II)

$$\mathsf{E}\left(\begin{array}{c}\dfrac{\begin{array}{c}\mathcal{D}_1 : \Gamma \vdash M@\Delta_1 : \sigma_1 \\ \mathcal{D}_2 : \Gamma \vdash M@\Delta_2 : \sigma_2\end{array}}{\Gamma \vdash M@(\Delta_1 \wedge \Delta_2) : \sigma_1 \wedge \sigma_2}\ (\wedge I)\end{array}\right) \triangleq \left\{\begin{array}{l}\dfrac{\begin{array}{c}\mathsf{E}(\mathcal{D}_1) : \Gamma' \vdash_\wedge M' : \sigma_1 \\ \mathsf{E}(\mathcal{D}_2) : \Gamma' \vdash_\wedge M' : \sigma_2\end{array}}{\Gamma' \vdash_\wedge M' : \sigma_1 \wedge \sigma_2}\ (\wedge I) \\ \mathcal{J}(\Gamma) \equiv \Gamma' \ \&\ \mathcal{J}(M@(\Delta_1 \wedge \Delta_2)) \equiv M'\end{array}\right.$$

$$\mathsf{E}\left(\dfrac{\mathcal{D} : \Gamma \vdash M@\Delta : \sigma_1 \wedge \sigma_2}{\Gamma \vdash M@(\swarrow\Delta) : \sigma_1}\ (\wedge E_L)\right) \triangleq \left\{\begin{array}{l}\dfrac{\mathsf{E}(\mathcal{D}) : \Gamma' \vdash_\wedge M' : \sigma_1 \wedge \sigma_2}{\Gamma' \vdash_\wedge M' : \sigma_1}\ (\wedge E_L) \\ \mathcal{J}(\Gamma) \equiv \Gamma' \ \&\ \mathcal{J}(M@\Delta) \equiv M'\end{array}\right.$$

$$\mathsf{E}\left(\dfrac{\mathcal{D} : \Gamma \vdash M@\Delta : \sigma_1 \wedge \sigma_2}{\Gamma \vdash M@(\Delta\searrow) : \sigma_2}\ (\wedge E_R)\right) \triangleq \left\{\begin{array}{l}\dfrac{\mathsf{E}(\mathcal{D}) : \Gamma' \vdash_\wedge M' : \sigma_1 \wedge \sigma_2}{\Gamma' \vdash_\wedge M' : \sigma_2}\ (\wedge E_R) \\ \mathcal{J}(\Gamma) \equiv \Gamma' \ \&\ \mathcal{J}(M@\Delta) \equiv M'\end{array}\right.$$

The Function D (I)

$$\mathsf{D}\left(\frac{x{:}\sigma \in \Gamma'}{\Gamma' \vdash_\wedge x : \sigma}\ (Var) \right) \triangleq \begin{cases} \dfrac{x@\iota{:}\sigma \in \Gamma}{\Gamma \vdash x@\iota : \sigma}\ (Var) \\ \mathcal{J}(\Gamma) \equiv \Gamma' \end{cases}$$

$$\mathsf{D}\left(\frac{\mathcal{D} : \Gamma', x{:}\sigma_1 \vdash_\wedge M' : \sigma_2}{\Gamma' \vdash_\wedge \lambda x.M' : \sigma_1 \to \sigma_2}\ (\to I) \right) \triangleq \begin{cases} \dfrac{\mathsf{D}(\mathcal{D}) : \Gamma, x@\iota{:}\sigma_1 \vdash M@\Delta : \sigma_2}{\Gamma \vdash (\lambda x@\iota.M)@(\lambda\iota{:}\sigma_1.\Delta) : \sigma_1 \to \sigma_2}\ (\to I) \\ \mathcal{J}(\Gamma, x@\iota{:}\sigma_1) \equiv \Gamma', x{:}\sigma_1 \\ \mathcal{J}(M@\Delta) \equiv M' \end{cases}$$

$$\mathsf{D}\left(\frac{\begin{array}{c}\mathcal{D}_1 : \Gamma' \vdash_\wedge M' : \sigma_1 \to \sigma_2 \\ \mathcal{D}_2 : \Gamma' \vdash_\wedge N : \sigma_1\end{array}}{\Gamma' \vdash_\wedge M'\,N' : \sigma_2}\ (\to E) \right) \triangleq \begin{cases} \dfrac{\begin{array}{c}\mathsf{D}(\mathcal{D}_1) : \Gamma \vdash M@\Delta_1 : \sigma_1 \to \sigma_2 \\ \mathsf{D}(\mathcal{D}_2) : \Gamma \vdash N@\Delta_2 : \sigma_1\end{array}}{\Gamma \vdash (M\,N)@(\Delta_1\,\Delta_2) : \sigma_2}\ (\to E) \\ \mathcal{J}(\Gamma) \equiv \Gamma' \\ \mathcal{J}(M@\Delta_1) \equiv M' \ \&\ \mathcal{J}(N@\Delta_2) \equiv N' \end{cases}$$

The Function D (II)

$$\mathsf{D}\left(\frac{\begin{array}{c}\mathcal{D}_1 : \Gamma' \vdash_\wedge M' : \sigma_1 \\ \mathcal{D}_2 : \Gamma' \vdash_\wedge M' : \sigma_2\end{array}}{\Gamma' \vdash_\wedge M' : \sigma_1 \wedge \sigma_2}\ (\wedge I)\right) \triangleq \begin{cases} \dfrac{\begin{array}{c}\mathsf{D}(\mathcal{D}_1) : \Gamma \vdash M@\Delta_1 : \sigma_1 \\ \mathsf{D}(\mathcal{D}_2) : \Gamma \vdash M@\Delta_2 : \sigma_2\end{array}}{\Gamma \vdash M@(\Delta_1 \wedge \Delta_2) : \sigma_1 \wedge \sigma_2}\ (\wedge I) \\ \mathcal{J}(\Gamma) \equiv \Gamma' \ \&\ \mathcal{J}(M@(\Delta_1 \wedge \Delta_2)) \equiv M' \end{cases}$$

$$\mathsf{D}\left(\frac{\mathcal{D} : \Gamma' \vdash_\wedge M' : \sigma_1 \wedge \sigma_2}{\Gamma' \vdash_\wedge M' : \sigma_1}\ (\wedge E_L)\right) \triangleq \begin{cases} \dfrac{\mathsf{D}(\mathcal{D}) : \Gamma \vdash M@\Delta : \sigma_1 \wedge \sigma_2}{\Gamma \vdash M@(\swarrow\Delta) : \sigma_1}\ (\wedge E_L) \\ \mathcal{J}(\Gamma) \equiv \Gamma' \ \&\ \mathcal{J}(M@\Delta) \equiv M' \end{cases}$$

$$\mathsf{D}\left(\frac{\mathcal{D} : \Gamma' \vdash_\wedge M' : \sigma_1 \wedge \sigma_2}{\Gamma' \vdash_\wedge M' : \sigma_2}\ (\wedge E_R)\right) \triangleq \begin{cases} \dfrac{\mathsf{D}(\mathcal{D}) : \Gamma \vdash M@\Delta : \sigma_1 \wedge \sigma_2}{\Gamma \vdash M@(\Delta\searrow) : \sigma_2}\ (\wedge E_R) \\ \mathcal{J}(\Gamma) \equiv \Gamma' \ \&\ \mathcal{J}(M@\Delta) \equiv M' \end{cases}$$

INTER:

Some Theorems

Galleria

(Isomorphism) The system $\Lambda\!\!\wedge$ and $\Lambda^{\mathsf{u}}\!\wedge$ are isomorphic

(Unicity of Typing) If $\Gamma \vdash P@\Delta : \sigma$, and $\Gamma \vdash P@\Delta : \sigma'$, then $\sigma \equiv \sigma'$

(Subject Reduction) If $\Gamma \vdash P@\Delta : \sigma$, and $P \rightarrow_\beta Q$, then exists Δ' such that $\Gamma \vdash Q@\Delta' : \sigma$ with $\Delta \mapsto\!\!\!\twoheadrightarrow_{\iota\pi} \Delta'$

(Strong Normalization) $\Gamma \vdash M@\Delta : \sigma$ if and only if M is strongly normalizing

(Decidability of Typing for $\Lambda\!\!\wedge$) For a given M, the fwing problems are decidable:

1. Type Reconstruction: for a given Γ, and a type-store Δ is there a type σ such that $\text{Type}_\wedge(\Gamma, M@\Delta) = \sigma$?
2. Type Checking: given a context Γ, and a type-store Δ, and a type σ, is it true that $\text{Type}_\wedge(\Gamma, M@\Delta) = \sigma$?

(Soundness of $\text{Type}_\wedge$**)** For a closed M, and a given Δ, if $\text{Type}_\wedge(\varepsilon, M@\Delta) = \sigma$, then $\varepsilon \vdash M@\Delta : \sigma$ is derivable

INTER:

Algorithms

Type Reconstruction / Type Checking

$$\text{Type}_\wedge(\Gamma, M@\Delta) \quad \triangleq \quad \text{match } (M@\Delta) \text{ with}$$

$$(\_@(\swarrow\Delta_1)) \quad \Rightarrow \quad \sigma_1 \quad \text{if Type}_\wedge(\Gamma, M@\Delta_1) = \sigma_1 \wedge \sigma_2$$

$$(\_@(\Delta_1\searrow)) \quad \Rightarrow \quad \sigma_2 \quad \text{if Type}_\wedge(\Gamma, M@\Delta_1) = \sigma_1 \wedge \sigma_2$$

$$(\_@(\Delta_1 \wedge \Delta_2)) \quad \Rightarrow \quad \sigma_1 \wedge \sigma_2 \quad \text{if Type}_\wedge(\Gamma, M@\Delta_1) = \sigma_1 \text{ and Type}_\wedge(\Gamma, M@\Delta_2$$

$$(x@\_) \quad \Rightarrow \quad \sigma \quad \text{if } x@\iota{:}\sigma \in \Gamma$$

$$((\lambda x@\iota.M_1)@(\lambda\iota{:}\sigma_1.\Delta_1)) \quad \Rightarrow \quad \sigma_1{\to}\sigma_2 \quad \text{if Type}_\wedge((\Gamma, x@\iota{:}\sigma_1), M_1@\Delta_1) = \sigma_2$$

$$((M_1\ M_2)@(\Delta_1\ \Delta_2)) \quad \Rightarrow \quad \sigma_2 \quad \text{if Type}_\wedge(\Gamma, M_1@\Delta_1) \equiv \sigma_1{\to}\sigma_2 \text{ and Type}_\wedge(\Gamma, M$$

$$(\_@\_) \quad \Rightarrow \quad \text{false} \quad \text{otherwise}$$

$$\text{Typecheck}_\wedge(\Gamma, M@\Delta, \sigma) \quad \triangleq \quad \text{if Type}_\wedge(\Gamma, M@\Delta) = \sigma \text{ then true else false}$$