Natural Deduction via Graphs

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- Gentzen-Prawitz style natural deduction and Fitch style flag deduction

- Deduction graphs: definition and examples

- Cut-elimination in deduction graphs

- Term interpretations in simple type theory

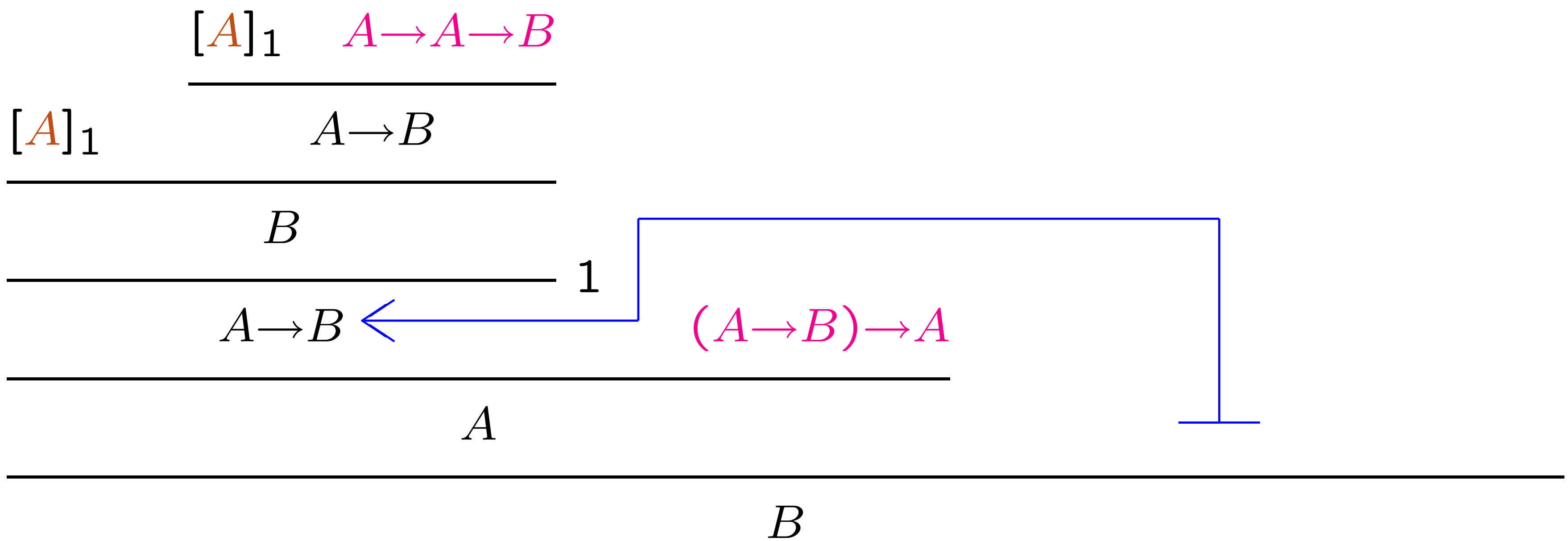
- Term interpretation into λ-let calculus

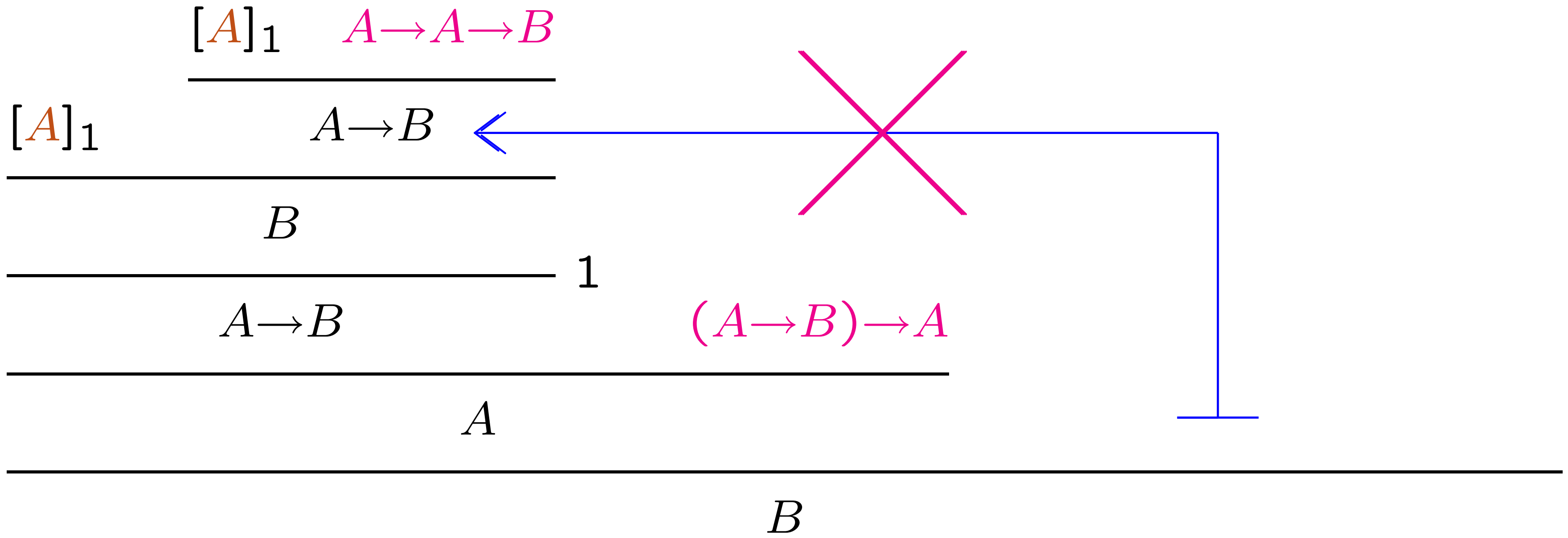
Gentzen-Prawitz style natural deduction

A proof of $A{\rightarrow}A{\rightarrow}B, (A{\rightarrow}B){\rightarrow}A \vdash B$.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        [A]_1 \quad \dfrac{[A]_1 \quad A{\rightarrow}A{\rightarrow}B}{A{\rightarrow}B}
      }{B}
    }{A{\rightarrow}B}\,1
    \qquad (A{\rightarrow}B){\rightarrow}A
  }{A}
  \qquad
  \dfrac{
    \dfrac{
      [A]_1 \quad \dfrac{[A]_1 \quad A{\rightarrow}A{\rightarrow}B}{A{\rightarrow}B}
    }{B}
  }{A{\rightarrow}B}\,1
}{B}
$$

The subproof of $A{\rightarrow}A{\rightarrow}B \vdash A{\rightarrow}B$ occurs twice.

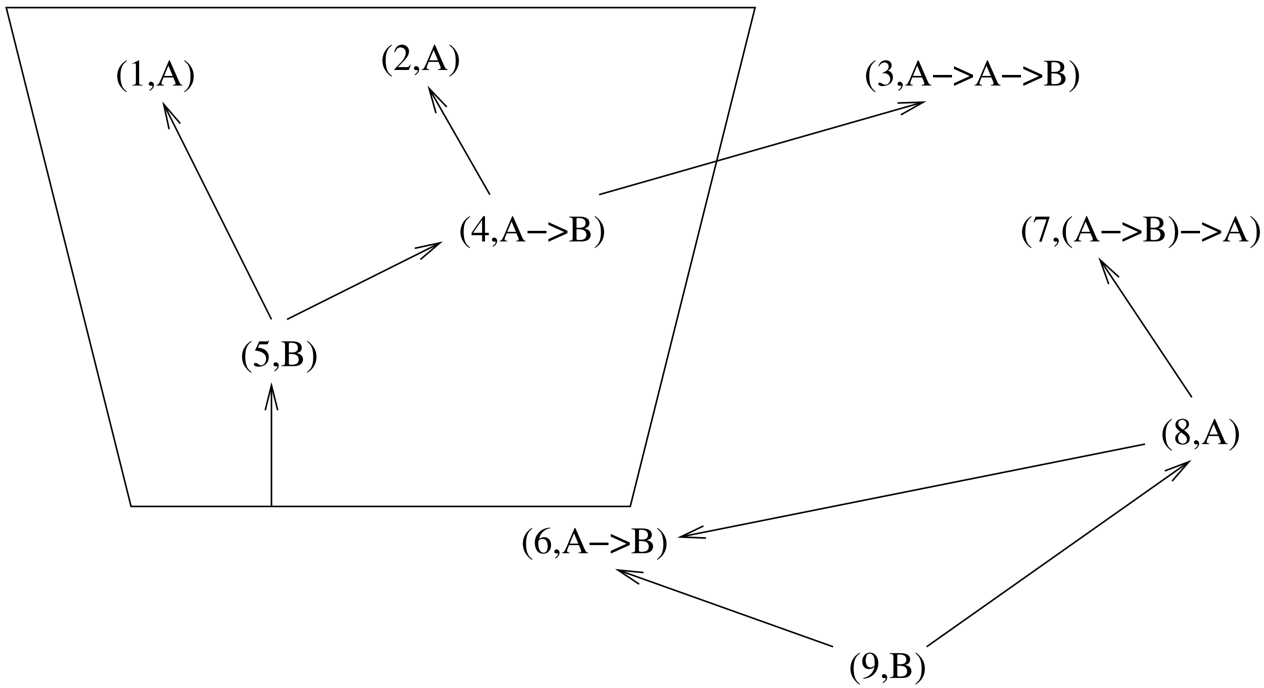

[A]1
A→A→B
[A]1
A→B
B
1
A→B
(A→B)→A
A
B


[A]1
A→A→B
[A]1
A→B
B
1
A→B
(A→B)→A
A
B

Fitch style natural deduction of $A \rightarrow A \rightarrow B, (A \rightarrow B) \rightarrow A \vdash B$.

1	$A \rightarrow A \rightarrow B$	
2	$(A \rightarrow B) \rightarrow A$	
3	A	
4	$A \rightarrow B$	$\rightarrow$E, 1, 3
5	B	$\rightarrow$E, 4, 3
6	$A \rightarrow B$	$\rightarrow$I, 3, 5
7	A	$\rightarrow$E, 2, 6
8	B	$\rightarrow$E, 6, 7

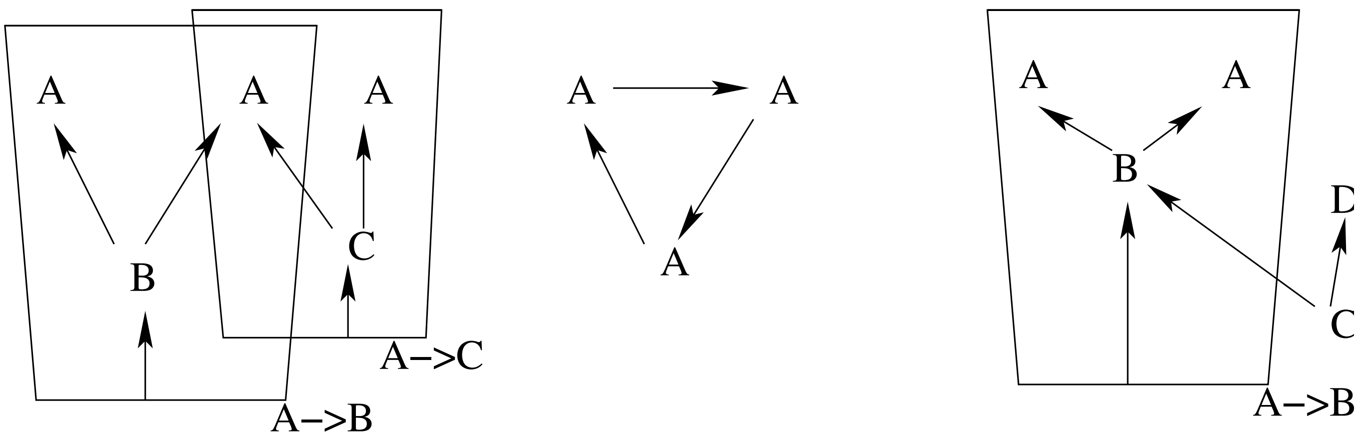
Deduction graph of $A \to A \to B, (A \to B) \to A \vdash B$.



Deduction graph of $A \to A \to B, (A \to B) \to A \vdash B$.

What is a (good definition of) deduction graph?

What we want to forbid:



We consider closed box directed graphs:

$\langle X, G, (\mathcal{B}_i)_{i \in I} \rangle$ with

- X a set of labels
- G is a directed graph where all nodes have a label in X
- $(\mathcal{B}_i)_{i \in I}$ is a collection of boxes: sets of nodes of G; each box is a node itself,

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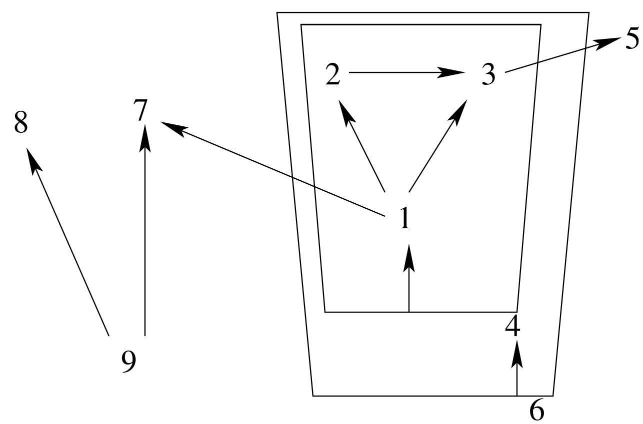
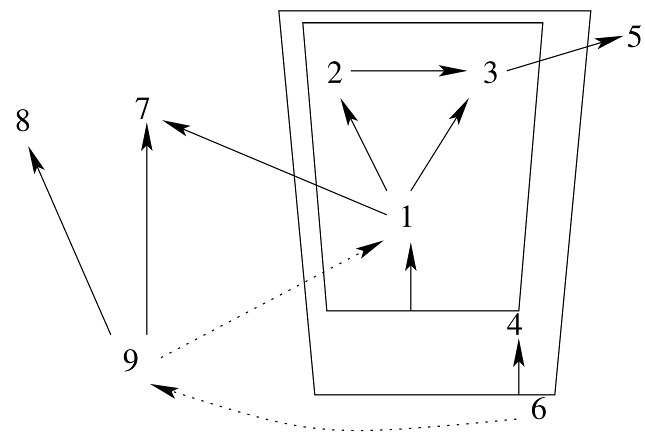
that should satisfy:

- (Non-overlap) Two boxes are disjoint or one is contained in the other.
- (Box-node edge) Only one outgoing edge from a box-node, pointing into the box itself.
- (No edges into a box) Apart from the edge from the box-node, there are no edges pointing into a box.

A well-formed and a non-well-formed closed box directed graph



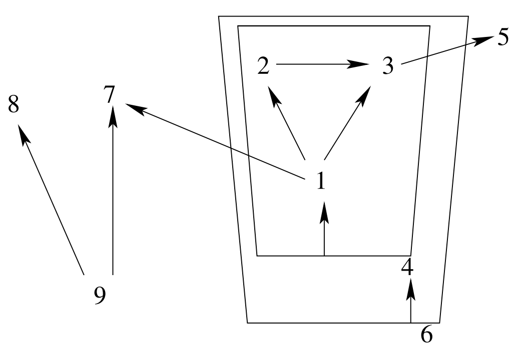


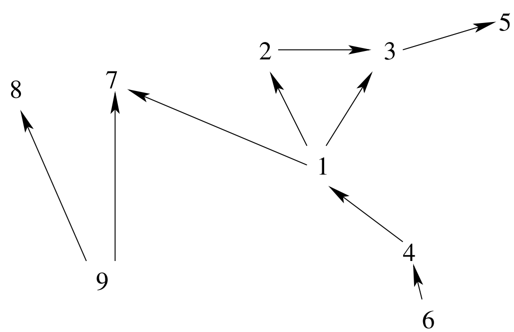
Notions:

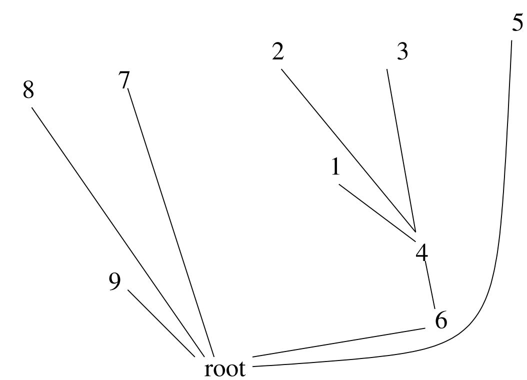
- top level node
- free node
- node n is in scope of node p (def: p is in all boxes that n is in [“p may point to n”])

Reminiscence: bigraphs of Milner, where the link graph describes the nodes and edges and the place graph describes the nesting of nodes.

We can view a closed box directed graphs as a combination of link graph and a place graph:







The collection of deduction graphs for minimal proposition logic is defined inductively.

Deduction graphs will be closed box-directed graphs.

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Deduction graphs will be closed box-directed graphs.

- **Axiom** A single node (n, A) is a deduction graph,
- **Join** If G and G' are disjoint deduction graphs, then $G'' := G \cup G'$ is a deduction graph.
- **$\to$-E** If G is a deduction graph with nodes $(n, A \to B)$ and (m, A) at the top level, then the graph $G' := G$ with
    - · a new node (p, B) at the top level
    - · an edge $(p, B) \longrightarrow\!\!\triangleright (n, A \to B)$,
    - · an edge $(p, B) \longrightarrow\!\!\triangleright (m, A)$,

is a deduction graph.

Definition of the collection of deduction graphs ctd.

• **$\to$-I** If G is a deduction graph with a node (j, B) and a finite set of free nodes with label A, $(n_1, A), \ldots, (n_k, A)$, all at the top level, then the graph $G' := G$ with

· A box $\mathcal{B}$ with box-node $(n, A \to B)$, containing the nodes (j, B) and (n_1, A), $\ldots$, (n_k, A) and no other nodes that were free in G,

· An edge $(n, A \to B) \longrightarrow\!\triangleright (j, B)$

is a deduction graph under the proviso that it is a well-formed closed box directed graph

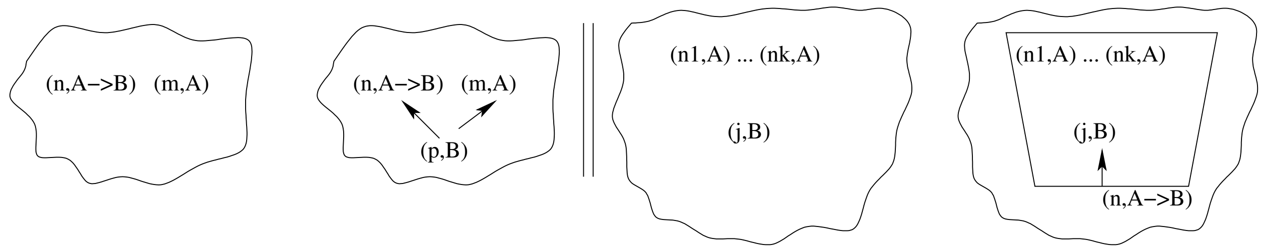
• **Repeat** If G is a deduction graph containing a node (n, A) at the top level, the graph $G' := G$ with

· a new node (m, A) at the top level,

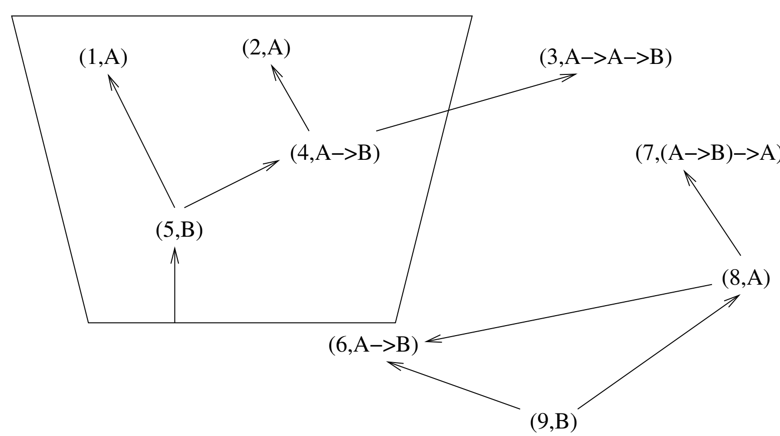
· an edge $(m, A) \longrightarrow\!\triangleright (n, A)$

is a deduction graph.

The →-E and →-I rule of deduction graphs



Example of a deduction graph



Lemma G is a deduction graph if and only if the following hold

- G is a finite closed box directed graph,
- every node of G is of one of the following four types:

A n has no outgoing edges

E n has label B and has exactly two outgoing edges to nodes in scope of n: $(n, B) \longrightarrow\!\triangleright (m, A{\to}B)$ and $(n, B) \longrightarrow\!\triangleright (p, A)$

I n is a node of a box $\mathcal{B}$ with label $A{\to}B$ and has exactly one outgoing edge, into the box: $(n, A{\to}B) \longrightarrow\!\triangleright (j, B)$. All 'free' nodes inside the box have label A.

R n has label A and has exactly one outgoing edge to a node in scope of n: $(n, A) \longrightarrow\!\triangleright (m, A)$.

- there is a box-topological ordering on G.

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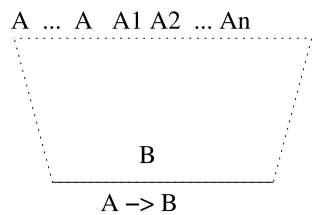
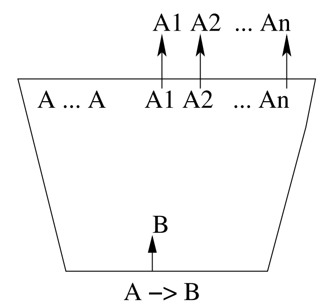
- there is a box-topological ordering on G.

Definition: A box-topological ordering $>$ is an ordering of the nodes that respects the place graph and the link graph.
(So $n > p$ if $(n, A) \longrightarrow\!\triangleright (p, B)$ or p is in a box with box node n.)

Lemma Every natural deduction is a deduction graph and every Fitch style flag deduction is a deduction graph.

Natural deduction:

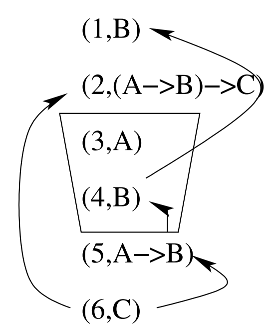




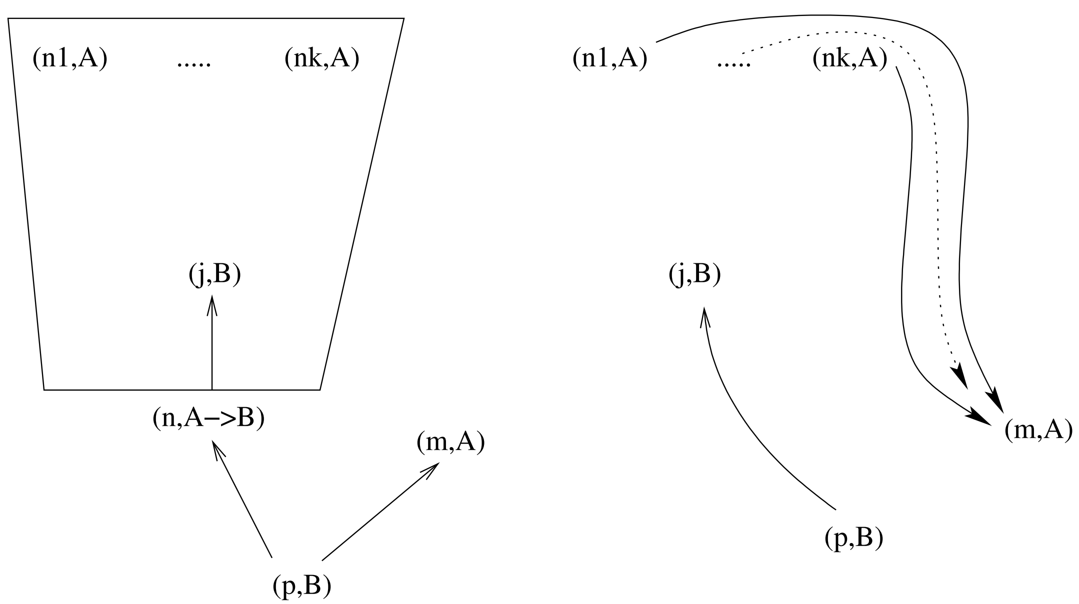
Fitch deduction:

1	B	
2	(A–>B)–>C	
3	A	Flag
5	A–>B	–>I 3,1
6	C	–>E 2,5

1	B	
2	(A–>B)–>C	
3	A	Flag
4	B	Repeat 1
5	A–>B	–>I 3,4
6	C	–>E 2,5



Cut elimination



Definition A safe cut in G is a subgraph consisting of

- A box-node $(n, A{\rightarrow}B)$,
- A node (p, B),
- A node (m, A),
- edges $(p, B) \longrightarrow\!\triangleright (n, A{\rightarrow}B)$ and $(p, B) \longrightarrow\!\triangleright (m, A)$

such that

- the edge $(p, B) \longrightarrow\!\triangleright (n, A{\rightarrow}B)$ is the unique edge to $(n, A{\rightarrow}B)$,
- the node (m, A) is in scope of $(n, A{\rightarrow}B)$.

Safe cuts can be contracted by removing the box and the box node and rearranging (and adding) edges.

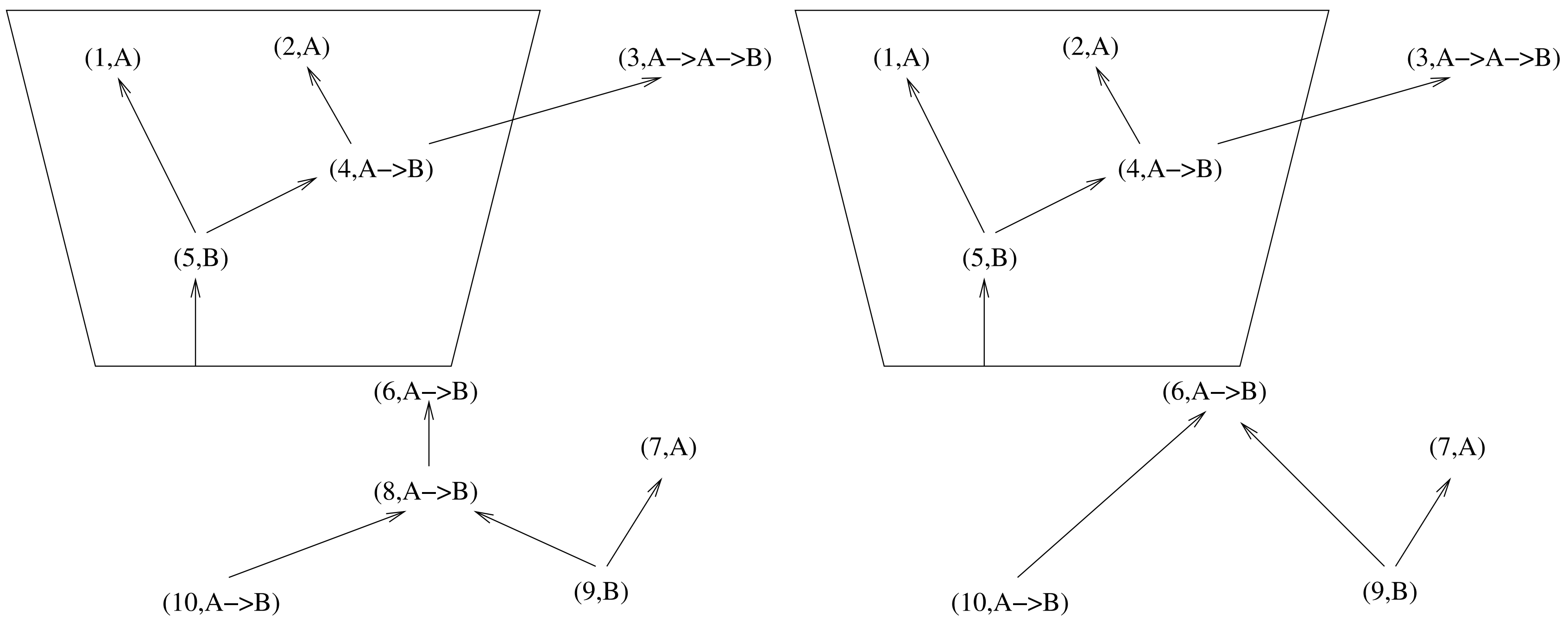
Definition A cut in G is a subgraph consisting of

- A box-node $(n, A \rightarrow B)$,
- A node (p, B),
- A node (m, A),
- A sequence of R-nodes $(s_0, A \rightarrow B), \ldots, (s_i, A \rightarrow B)$,
- Edges $(p, B) \longrightarrow\!\triangleright (s_i, A \rightarrow B) \longrightarrow\!\triangleright \ldots \longrightarrow\!\triangleright (s_0, A \rightarrow B) \longrightarrow\!\triangleright (n, A \rightarrow B)$,
- An edge $(p, B) \longrightarrow\!\triangleright (m, A)$.

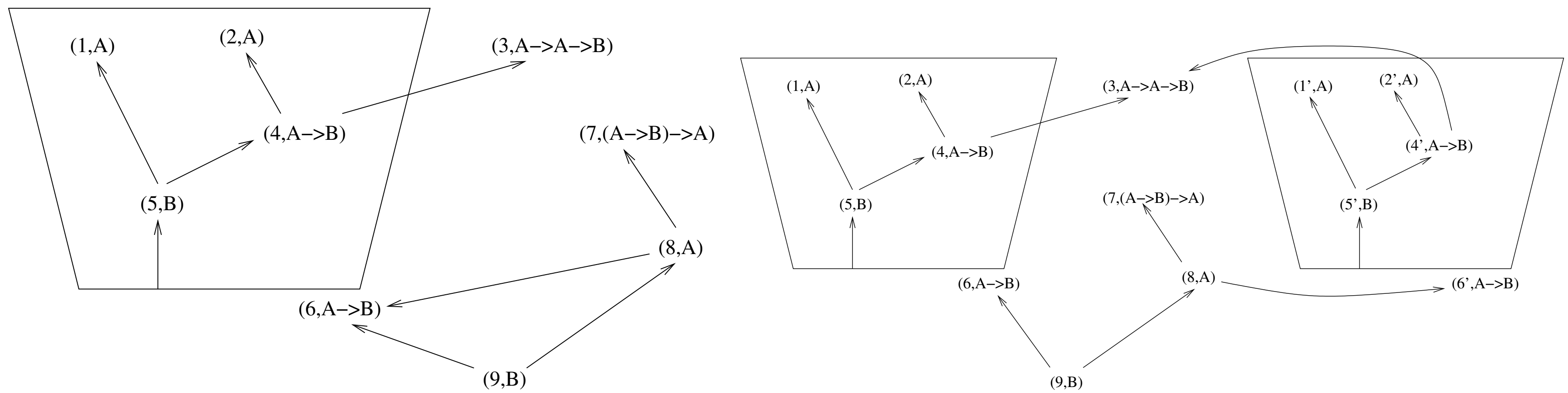
There are three possible ways in which the contraction of a cut can be obstructed.

- Cut hidden by a repeat
- Cut hidden by sharing
- Cut hidden by a depth conflict

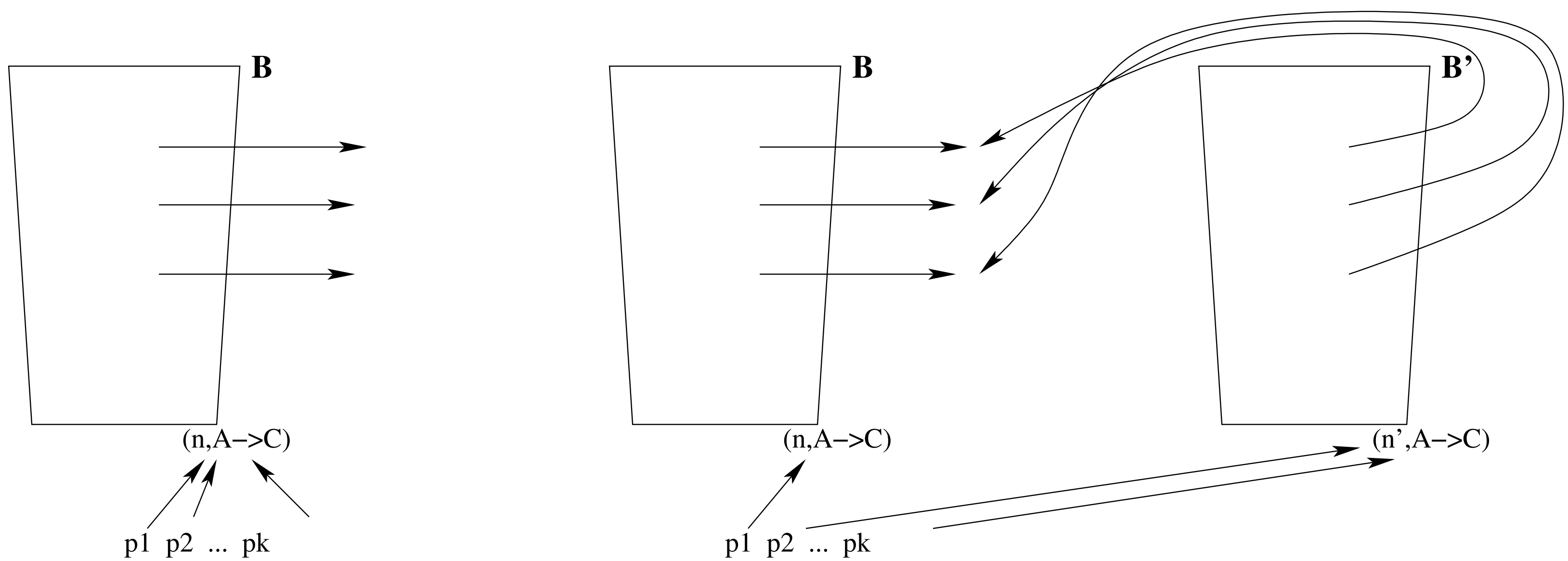
Deduction graph with a cut hidden by a repeat and the same graph with the cut made explicit.



Deduction graph with a cut hidden by sharing and the same graph unshared

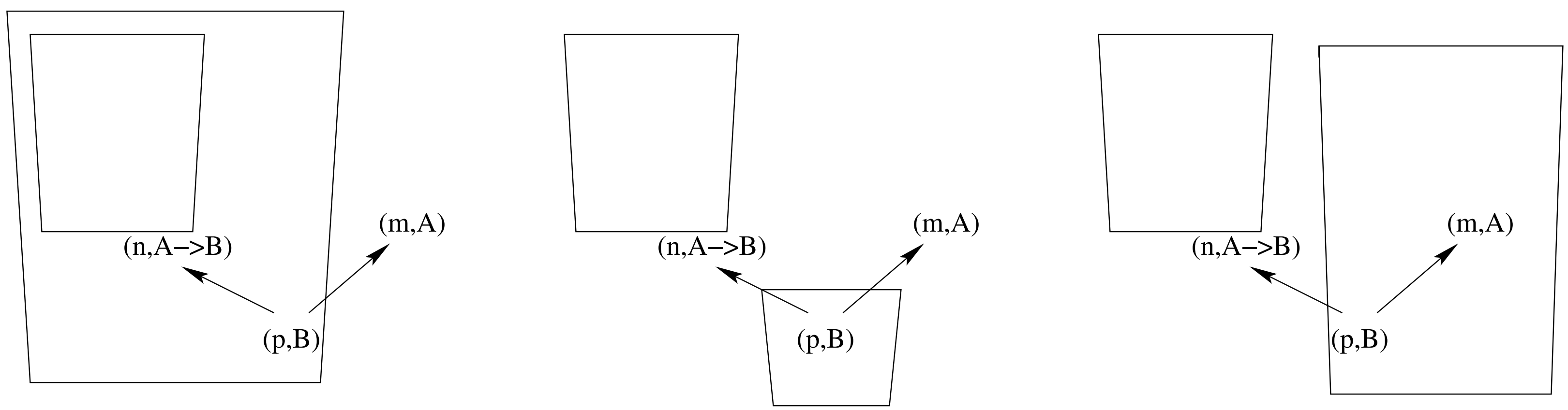


General process of unsharing

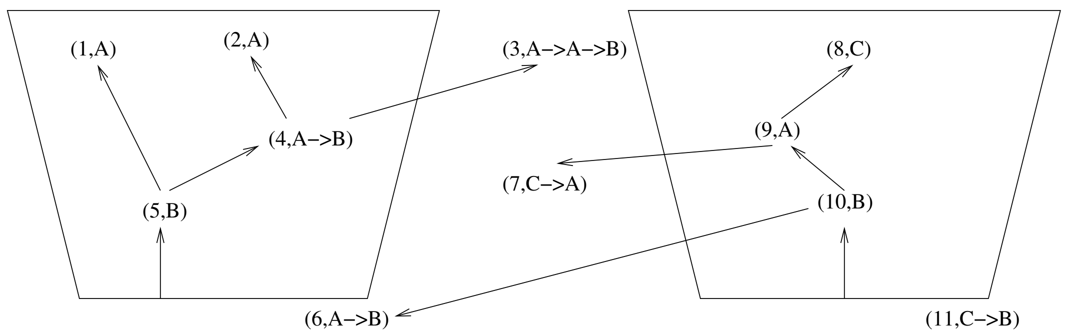


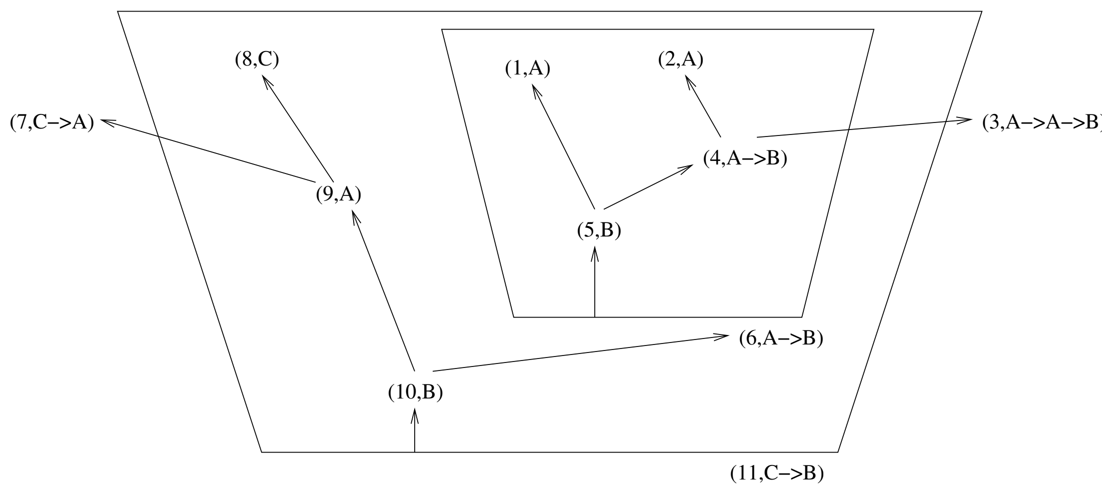
Cuts hidden by a depth conflict

Three possible positions of the cut with respect to the boxes. The third presents a depth conflict.



Cut hidden by a depth conflict and the cut made explicit via an incorporation.





The process of eliminating the cut c is the following.

- Make c explicit by removing repeats
- Make c explicit by unsharing
- Make c explicit by incorporation steps
- If c is safe, perform the cut-elimination step.

Computational content: from deduction graphs to λ-terms

Definition Given G with node n, define $[\![G,n]\!]$:

[A] If (n, A) has no outgoing edges, $[\![G,n]\!] := x_n^A$,

[E] If $(n, B) \longrightarrow\!\triangleright (m, A{\to}B)$, and $(n, B) \longrightarrow\!\triangleright (p, A)$,

$[\![G,n]\!] := [\![G_m,m]\!]\,[\![G_p,p]\!]$,

[I] If $(n, A{\to}B)$ is a box-node with $(n, A{\to}B) \longrightarrow\!\triangleright (j, B)$ and the free nodes of the box are $(n_1, A), \ldots, (n_k, A)$,

$[\![G,n]\!] := \lambda x{:}A.([\![G_j,j]\!][x_{n_1} := x, \ldots, x_{n_k} := x])$,

[R] If $(n, A) \longrightarrow\!\triangleright (m, A)$, $[\![G,n]\!] := [\![G_m,m]\!]$

(NB G_m is the subgraph generated from m.)

Computational content: from deduction graphs to λ-terms

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[R] If $(n, A) \longrightarrow\!\triangleright (m, A)$, $[\![G, n]\!] := [\![G_m, m]\!]$

Properties:

- If G' is obtained from G by unsharing, incorporation or R-elimination, then $[\![G, n]\!] = [\![G', n]\!]$.
- If $G \twoheadrightarrow_{\text{cut}} G'$,then $[\![G, n]\!] \twoheadrightarrow_\beta [\![G', n]\!]$.

Strong Normalization of cut-elimination for deduction graphs.

The map $[\![G,n]\!]$ doesn't prove SN, because reductions may be "thrown away".
We need to preserve all structure.

We consider $\lambda{\rightarrow}\langle\rangle$, simple type theory with a tupling constructor:

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N_1 : \tau_1 \quad \ldots \quad \Gamma \vdash N_k : \tau_k}{\Gamma \vdash \langle M, N_1, \ldots, N_k \rangle : \sigma}$$

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$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N_1 : \tau_1 \quad \ldots \quad \Gamma \vdash N_k : \tau_k}{\Gamma \vdash \langle M, N_1, \ldots, N_k\rangle : \sigma}$$

$$\begin{array}{rcl}
(\lambda x{:}\sigma.M)N & \longrightarrow_{\bar{\beta}} & M[x := N] \text{ if } x \in \mathsf{FV}(M) \\
(\lambda x{:}\sigma.M)N & \longrightarrow_{\bar{\beta}} & \langle M, N\rangle \text{ if } x \notin \mathsf{FV}(M) \\
\langle M, P_1, \ldots, P_k\rangle N & \longrightarrow_{\bar{\beta}} & \langle MN, P_1, \ldots, P_k\rangle \\
\langle \ldots, \langle M, P_1, \ldots, P_k\rangle, N_1, \ldots, N_p\rangle & \longrightarrow_{\bar{\beta}} & \langle \ldots, M, P_1, \ldots, P_k, N_1, \ldots, N_p\rangle
\end{array}$$

Definition Given G with node n, define $\langle\!\![G,n]\!\!\rangle$ similar to $[\![G,n]\!]$ except for:

• [I] If $(n, A{\rightarrow}B)$ is a box-node with $(n, A{\rightarrow}B) \longrightarrow\!\triangleright (j, B)$, the free nodes of the box are $n_1, \ldots, n_k$ and the nodes without incoming edges inside the box are $m_1, \ldots, m_p$, then

$$\langle\!\![G,n]\!\!\rangle := \lambda x{:}A.\langle \langle\!\![G,j]\!\!\rangle, \langle\!\![G,m_1]\!\!\rangle, \ldots, \langle\!\![G,m_p]\!\!\rangle \rangle [x_{n_1} := x, \ldots, x_{n_k} := x].$$

The interpretation of the whole deduction graph G, $\langle\!\![G]\!\!\rangle$, is

$$\langle \langle\!\![G,r_1]\!\!\rangle, \ldots, \langle\!\![G,r_l]\!\!\rangle \rangle$$

where $r_1, \ldots, r_l$ are the top nodes without incoming edges of G.

Lemma (Cut-elimination is $\bar{\beta}$-reduction in $\lambda\!\rightarrow\!\langle\rangle$)

$$\text{If } G \longrightarrow_{\mathsf{cut}} G', \text{ then } \langle\!\![G]\!\!\rangle \longrightarrow\!\!\!\!\gg^{+}_{\bar{\beta}} \langle\!\![G']\!\!\rangle.$$

Corollary: cut-elimination is SN (because $\longrightarrow_{\bar{\beta}}$ is SN).

$\langle\!\langle G \rangle\!\rangle$ preserves reduction paths, but it does not preserve all structure:

If G' is obtained from G by unsharing, incorporation or R-elimination, then $\langle\!\langle G \rangle\!\rangle \equiv_p \langle\!\langle G' \rangle\!\rangle$.

We introduce an interpretation $\langle\!\langle G \rangle\!\rangle$ of deduction graphs as $\lambda{\rightarrow} + \mathsf{let}$-contexts that preserves all structure (e.g. binding)
($\lambda{\rightarrow} + \mathsf{let}$ = simple typed λ calculus with lets)

Contexts of $\lambda{\to} + \mathsf{let}$:

$$\begin{array}{lrcl} \text{Terms} & T & := & x \mid (xy) \mid \lambda x.C[y] \\ \text{Contexts} & C & := & [-]_l \mid \mathsf{let}\, x = T \,\mathsf{in}\, C[-] \end{array}$$

There are typing rules for terms and well-formedness rules for contexts.

For a deduction graph G and a box-topological ordering φ of G, we define the $\lambda{\to} + \mathsf{let}$ context $\langle\!\langle G \rangle\!\rangle_\varphi$.

The ordering φ plays a crucial role, but we suppress it in the notation.

Definition Let a deduction graph G and a box-topological ordering φ of G be given. $\langle\!\langle G \rangle\!\rangle$ is defined as follows.

• If G has only one node, $\langle\!\langle G \rangle\!\rangle = [-]$.

• If $|G| > 1$ let the φ-maximal element be n. Case distinction on what type of node n is:

[A] $\langle\!\langle G \rangle\!\rangle = \langle\!\langle G \setminus n \rangle\!\rangle$

[I] with $(n, A{\rightarrow}B) \longrightarrow\!\triangleright (j, B)$ with associated box $\mathcal{B}$, then

$\langle\!\langle G \rangle\!\rangle = \langle\!\langle G \setminus \mathcal{B} \rangle\!\rangle[\mathsf{let}\, x_n = (\lambda x_m{:}A.\langle\!\langle \mathcal{B}^m \rangle\!\rangle[x_j])\,\mathsf{in}\,[-]]$,

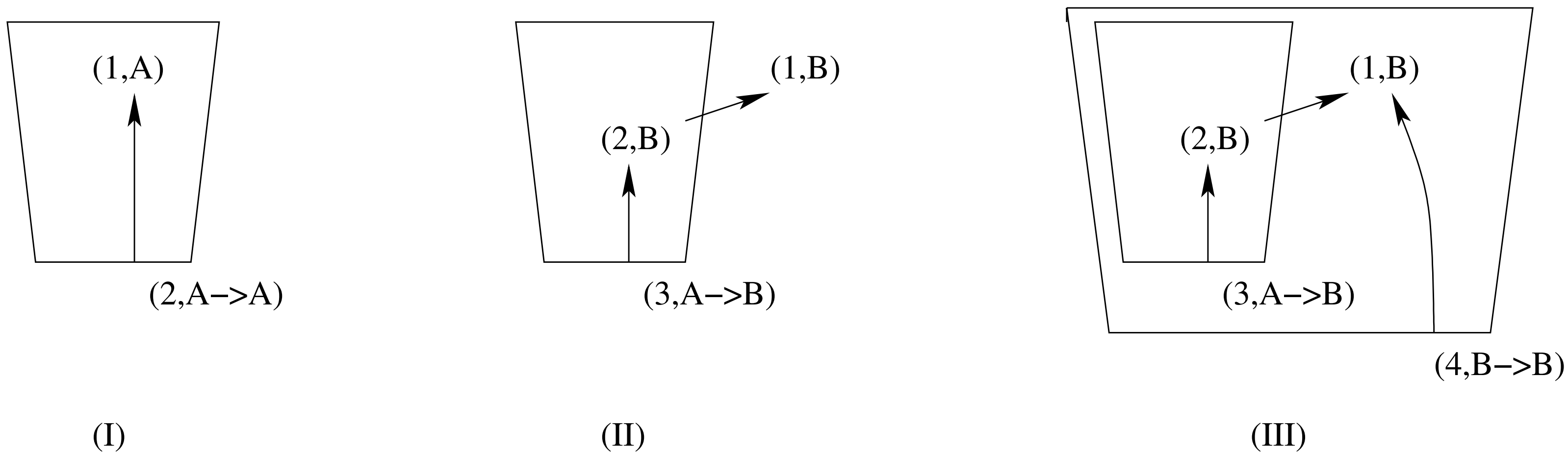
[E] with $(n, B) \longrightarrow\!\triangleright (i, A{\rightarrow}B)$ and $(n, B) \longrightarrow\!\triangleright (j, A)$, then

$\langle\!\langle G \rangle\!\rangle := \langle\!\langle G \setminus n \rangle\!\rangle[\mathsf{let}\, x_n = (x_i x_j)\,\mathsf{in}\,[-]]$

[R] with $n \longrightarrow\!\triangleright l$, then

$\langle\!\langle G \rangle\!\rangle = \langle\!\langle G \setminus n \rangle\!\rangle[\mathsf{let}\, x_n = x_l\,\mathsf{in}\,[-]]$

Three simple examples



The translations of these deduction graphs are:

$$\mathsf{let}\, x_2 = (\lambda y{:}A.\mathsf{let}\, x_1 = y \,\mathsf{in}\, x_1) \,\mathsf{in}\, [-]$$
$$\mathsf{let}\, x_3 = (\lambda y{:}A.\mathsf{let}\, x_2 = x_1 \,\mathsf{in}\, x_2) \,\mathsf{in}\, [-]$$
$$\mathsf{let}\, x_4 = (\lambda z{:}B.\mathsf{let}\, x_1 = z \,\mathsf{in}\, \mathsf{let}\, x_3 = (\lambda y{:}A.\mathsf{let}\, x_2 = x_1 \,\mathsf{in}\, x_2) \,\mathsf{in}\, x_1) \,\mathsf{in}\, [-]$$

Rewriting for $\lambda{\rightarrow} + \mathsf{let}$-contexts (\$ denotes either '$-$' or a variable)

$$\mathsf{let}\, x = P \,\mathsf{in}\, C[y] \longrightarrow_{GB} C[y] \qquad \text{if } x \notin \mathsf{FV}(C[y])$$

$$\mathsf{let}\, x = \lambda z{:}A.D[y] \,\mathsf{in}\, \mathsf{let}\, x_p = (x x_m) \,\mathsf{in}\, C[\$] \longrightarrow_{B}$$
$$D[z := x_m][\mathsf{let}\, x_p = y \,\mathsf{in}\, C[\$]] \qquad \text{if } x_n \notin \mathsf{FV}(C[\$])$$

$$\mathsf{let}\, x = P \,\mathsf{in}\, \mathsf{let}\, y = \lambda z.Q \,\mathsf{in}\, C[\$] \longrightarrow_{CM}$$
$$\mathsf{let}\, y = \lambda z.(\mathsf{let}\, x = P \,\mathsf{in}\, Q) \,\mathsf{in}\, C[\$] \qquad \text{if } x \notin \mathsf{FV}(C[\$])$$

$$\mathsf{let}\, x = P \,\mathsf{in}\, C[\$][y := x] \longrightarrow_{CP}$$
$$\mathsf{let}\, y = g(P) \,\mathsf{in}\, \mathsf{let}\, x = P \,\mathsf{in}\, C[\$] \qquad \text{if } x, y \in \mathsf{FV}(C[\$])$$

$$\mathsf{let}\, y = x \,\mathsf{in}\, C[\$] \longrightarrow_{L} C[\$][y := x]$$

$$\mathsf{let}\, x = P \,\mathsf{in}\, \mathsf{let}\, y = Q \,\mathsf{in}\, C[\$] \simeq$$
$$\mathsf{let}\, y = Q \,\mathsf{in}\, \mathsf{let}\, x = P \,\mathsf{in}\, C[\$] \qquad \text{if } x \notin \mathsf{FV}(Q), y \notin \mathsf{FV}(P)$$

Lemma (Interpretation is independent of the box-topological ordering)
If G is a deduction graph and φ and ψ are both box-topological orderings of G, then

$$\langle\!\langle G \rangle\!\rangle_{\varphi} \simeq \langle\!\langle G \rangle\!\rangle_{\psi}$$

Lemma Reduction on deduction graphs corresponds to reduction in $\lambda{\rightarrow} + \mathsf{let}$

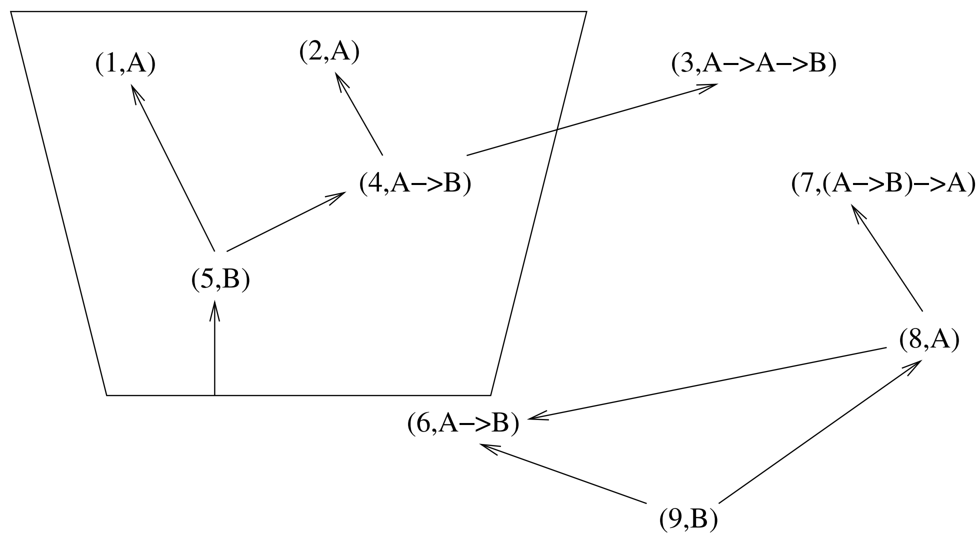
- If G' is obtained from G by an unsharing step, then $\langle\!\langle G\rangle\!\rangle \longrightarrow_{CP} \langle\!\langle G'\rangle\!\rangle$.

- If G' is obtained from G by an incorporation step, then $\langle\!\langle G\rangle\!\rangle \longrightarrow_{CM} \langle\!\langle G'\rangle\!\rangle$.

- If G' is obtained from G by a repeat elimination step, then $\langle\!\langle G\rangle\!\rangle \longrightarrow_{L} \langle\!\langle G'\rangle\!\rangle$.

- If G' is obtained from G by eliminating a safe cut, then $\langle\!\langle G\rangle\!\rangle \longrightarrow_{B} \langle\!\langle G'\rangle\!\rangle$.

Example



$$
\begin{aligned}
\langle\!\langle G \rangle\!\rangle \;&=\; \mathsf{let}\, x_6 = (\lambda y{:}A.\mathsf{let}\, x_1 = y, x_2 = y, x_4 = (x_3 y), x_5 = (x_4 y)\,\mathsf{in}\, x_5)\,\mathsf{in} \\
&\quad\; \mathsf{let}\, x_8 = (x_7 x_6)\,\mathsf{in}\,\mathsf{let}\, x_9 = (x_6 x_8)\,\mathsf{in}\,[-]
\end{aligned}
$$

Future work

- Study the reduction relations on $\lambda{\rightarrow} + \mathsf{let}$
- Study the correspondence with proof nets
- How flexible / generalizable is the notion of deduction graph?